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MANDYAM TONDANUR NARANIENGAR, 1870-1940.

IN the *Gazette* for July 1907, Professor V. Ramaswami Aiyar wrote : " It may perhaps interest you to know that I have formed a Mathematical Society here. It started with twenty members. . . . If the Society succeeds, its success will in no small measure be due to the *Mathematical Gazette*. . . ." One of the foundation members of the new Society was M. T. Naraniengar, who edited its periodical, the *Journal of the Indian Mathematical Society*, from its start in 1909 till June 1927. In the early days, his task can not have been light, his labours not inconsiderable. The drudgery of re-writing papers, re-drawing diagrams, so that compositor and draughtsman should be able to set about their work with confidence, the cares of proof-reading, made heavier by the inexperience, in those days, with which Indian printing faced the problems of setting up complicated mathematical formulae, must have made heavy demands on Naraniengar's patience and enthusiasm. These demands were met, to quote Dr. A. Narasinga Rao, with " intense devotion and an unlimited capacity for silent, unobtrusive, tireless work "—no bad equipment for an editor.

In 1933, the Society began the publication of a more elementary journal, *The Mathematics Student*, of which the number for March 1941 has just reached us. This is a Naraniengar Memorial Issue, and in it mathematicians from all India join to express sorrow at their loss, and pride in the ordered, noble life of a beloved colleague, whose influence on the development of mathematics in India has been so potent and so beneficial.

POINTS ISOGONALLY CONJUGATE WITH RESPECT TO A TRIANGLE.

By P. W. Wood.

1. *Definitions and properties.*

If L, M, N are the feet of the perpendiculars from P to the sides of a triangle ABC, the circle through L, M, N is the pedal circle of P with respect to the triangle ABC.

(i) There is a unique conic touching the sides of ABC and having P as one focus ; LMN is its auxiliary circle and the other focus P' of this conic is the isogonal conjugate of P with respect to the triangle ABC. P' is the centre of the circle through the reflexions of P in the sides of ABC.

(ii) P and P' have the same pedal circle with respect to ABC.

(iii) The two pairs of lines AP, AP' and AB, AC have the same angle bisectors, etc., so that P, P' are conjugate points with respect to all rectangular hyperbolas through the centres of the four circles touching the sides of ABC.

(iv) If L', M', N' are the feet of the perpendiculars from P' to the sides of ABC,

$$PL \cdot P'L' = PM \cdot P'M' = PN \cdot P'N'.$$

Any one of these four properties leads to the other three by elementary geometry ; the pair P, P' are mutually related and may in particular be the circumcentre and the orthocentre of ABC. The centre of a circle touching the sides of ABC coincides with its isogonal conjugate.

Specially, if the conic touching the sides of ABC is a parabola, P is on the circle ABC, its pedal circle degenerates into the Simson Line of P with the line at infinity, and this Simson Line is the tangent at the vertex of the parabola ; P' is then the point at infinity on the axis of the parabola in the direction perpendicular to this Simson Line.

Conversely, if P is on the circle ABC, P' is at infinity and the inscribed conic is a parabola with its focus at P ; and, if P' is at infinity, P is on the circle ABC and the inscribed conic is a parabola with its focus at P.

2. *Coordinates.*

The use of trilinear coordinates (x, y, z) referred to the triangle ABC immediately links up properties (i), (ii), (iv) ; if (α, β, γ), $(\alpha', \beta', \gamma')$ are the coordinates of P, P', property (i) gives as alternative forms for the envelope equation of the inscribed conic $fmn + gnl + hlm = 0$ and

$$\lambda(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C) + (\alpha l + \beta m + \gamma n)(\alpha' l + \beta' m + \gamma' n) = 0,$$

so that $\alpha\alpha' = \beta\beta' = \gamma\gamma'$, which is property (iv).

Any rectangular hyperbola through the points $(\pm 1, \pm 1, \pm 1)$ (in particular the angle bisectors of A, B or C) has equation $px^2+qy^2+rz^2=0$, if $p+q+r=0$; P, P' are conjugate points with respect to this rectangular hyperbola, if $p\alpha\alpha'+q\beta\beta'+r\gamma\gamma'=0$, and this gives property (iv) again. The arguments are reversible.

If one of the pair P, P' lies on the line $lx+my+nz=0$, the other lies on the conic $lyz+mzx+nxy=0$ (specially, if the line is the line at infinity, the conic is the circumcircle of ABC); if the line passes through the circumcentre of ABC the conic passes through the orthocentre and is therefore a rectangular hyperbola. Conversely the isogonal conjugates of points on any rectangular hyperbola through A, B, C lie on a diameter of the circumcircle of ABC; call this the *associated* diameter of the circle.

Using rectangular Cartesian coordinates, consider the rectangular hyperbola $xy=1$, any point P of which has coordinates $(t, 1/t)$, A, B, C being given by $t=a, b, c$. The coordinates (x, y) of the isogonal conjugate P' may be found directly from any one of the properties (i)-(iv); for brevity write

$$p=a+b+c,\ q=1/a+1/b+1/c,\ r=abc,$$

so that a, b, c are the roots of $t^3-pt^2+qrt-r=0$, and note that the interchange of the pairs x and y, t and $1/t$, p and q, r and $1/r$ leave all results unchanged; in particular $(-1/r, -r)$ is the orthocentre of ABC and $(1/r, r)$ is the fourth intersection of the rectangular hyperbola and the circle ABC.

Property (iv) requires the three expressions like

$$\frac{\{x+bcy-(b+c)\}\{t+bc/t-(b+c)\}}{1+b^2c^2}$$

to be equal; so

$$\frac{x+bcy-(b+c)}{(1+b^2c^2)(t-a)}=\frac{x+cay-(c+a)}{(1+c^2a^2)(t-b)}=\frac{x+aby-(a+b)}{(1+a^2b^2)(t-c)}$$

$$=\frac{x\Sigma a^2(b-c)}{\Sigma a^2(b-c)(1+b^2c^2)(t-a)}=\frac{y\Sigma(b-c)bc}{\Sigma(b-c)(1+b^2c^2)(t-a)}$$

$$=\frac{-\Sigma a(b^2-c^2)}{\Sigma a(b-c)(1+b^2c^2)(t-a)},$$

and after removing the factor $(b-c)(c-a)(a-b)$, we have

$$\frac{x}{p-t}=\frac{y}{r-qrt}=\frac{1}{1-rt}.$$

Alternatively we can evade the calculation of the Σ's by considering the three equations like

$$x/bc+y-(1/b+1/c)=\kappa(t-a)(bc+1/bc),$$

where κ is symmetrical in a, b, c; this equation in terms of a is

$$ax/r+y-q+1/a=\kappa(t-a)(a/r+r/a)$$
$$=\kappa t(a/r+r/a)-\kappa r-\kappa(pa-qr+r/a)/r$$

which must be an identity in a. Equating coefficients, we have

$$x=\kappa(t-p), \quad y-q=\kappa(q-r), \quad 1=\kappa(rt-1)$$

and therefore

$$x=\frac{p-t}{1-rt}, \quad y=\frac{q-1/t}{1-1/rt}.$$

This simple algebra can however be avoided by considering the $1-1$ correspondence between the points P on the rectangular hyperbola and the points P' on the associated diameter of the circle ABC, and this correspondence leads directly to simple geometrical results.

The asymptote $y=0$ cuts the chord BC with equation

$$x+bcy-(b+c)=0$$

at the point $(b+c, 0)$, the perpendicular to BC through this point has equation $x-y/bc-(b+c)=0$ and therefore passes through the point (p, r); it follows that (p, r) is on the circle ABC and has $y=0$ as its Simson Line and also that $(1/r, q)$ is on the circle ABC and has $x=0$ as its Simson Line. The points (p, r), $(1/r, q)$ are the ends of the associated diameter of the circle, and their isogonal conjugates on the rectangular hyperbola are given by $t=0, \infty$.

We now have four special pairs of isogonal conjugates:

P (*on the rectangular hyperbola*)	*Parameter*	P' (*on the associated diameter*)
∞ point on $x=0$	$t=0$	(p, r)
∞ point on $y=0$	$t=\infty$	$(1/r, q)$
orthocentre	$t=-1/r$	circumcentre
4th intersection with circle ABC	$t=1/r$	∞ point

Any three of these pairs will determine the $1-1$ correspondence between P and P'; the cross ratios of four points on the rectangular hyperbola, of the four parameters which give these points, of their four isogonal conjugates on the associated diameter, of the x (or y) coordinates of either set of four points are all equal. So, if (x, y) is the isogonal conjugate of $(t, 1/t)$,

$$(p, 1/r, \infty, x)=(0, \infty, 1/r, t)=(r, q, \infty, y),$$

giving

$$x=\frac{p-t}{1-rt}, \quad y=\frac{q-1/t}{1-1/rt}.$$

Note that the ends $(\pm t)$ of a diameter of the rectangular hyperbola are harmonically separated by the infinity points $(t=0, \infty)$ of the rectangular hyperbola and therefore their isogonal conjugates are harmonically separated by the ends of the associated diameter of the circle and are therefore inverse points with respect to the circle.

3. *Pedal Circles.*

Having found the coordinates of P' it is easy to verify that the pedal circle of P (and of P') passes through O, the centre of the rectangular hyperbola, but this well-known result can be proved without the use of coordinates ; *e.g.* :

(i) If the lines joining D to the circular points at infinity cut the rectangular hyperbola again at E, F, then O is the foot of the perpendicular from P to EF ; the six sides of the triangles ABC, PEF inscribed in a conic touch another conic whose focus is P ; the auxiliary circle of this conic is the pedal circle of P and it passes through O.

(ii) The circumcircle of a triangle of tangents to a parabola is known to pass through the focus of the parabola ; reciprocation of this result with respect to any point on the directrix of the parabola shows that, if A, B, C, P are four points on a rectangular hyperbola with centre O, the conic with one focus at P touching the sides of ABC also touches the line through O perpendicular to OP.

If we use this result we can prove at once without using coordinates that, if A, B, C are any three points on a rectangular hyperbola the asymptotes are Simson Lines with respect to ABC.

Since there is a unique rectangular hyperbola through any four points (which are not orthocentric), the pedal circle of each point with respect to the triangle formed by the other three will pass through the centre of this rectangular hyperbola and, if the four points are concyclic, the four Simson Lines are concurrent.

Feuerbach's Theorem is an immediate consequence of this property : notice first that any rectangular hyperbola through A, B, C will pass through the orthocentre of ABC and that the pedal circle of the orthocentre is the nine points circle, which will pass through the centre of the rectangular hyperbola. If P is any point its pedal circle with respect to ABC will have in common with the nine points circle the centre O of the rectangular hyperbola through A, B, C, P ; the other common point of the two circles will be the centre O' of the rectangular hyperbola through A, B, C, P', since P and P' have the same pedal circle. This pedal circle will touch the nine points circle, if O and O' coincide ; this will happen in general, if PP' is a diameter of the circle ABC and in particular, if P and P' themselves coincide, in which case P is the centre of a circle touching the sides of ABC.

One other application of the property will suffice : if P is a variable point on a fixed diameter of the circumcircle of a triangle ABC, its isogonal conjugate P' with respect to ABC lies on a fixed rectangular hyperbola whose centre lies on the pedal circle of P. If the fixed diameter meets the sides of ABC in X, Y, Z, the circles on AX, BY, CZ as diameters have therefore one common point on the nine points circle of ABC and Pascal's Theorem shows that the other common point lies on the circumcircle of ABC ; this result is due to Thébault.

4. *Metrical properties of a quadrangle A, B, C, D.*

It seems natural that, just as the metrical properties of a triangle involve its circumcentre, the metrical properties of a quadrangle should involve the centre of the unique rectangular hyperbola through them. The construction of the rectangular hyperbola is simple : the circle through the middle points of the sides of BCD and the circle through the middle points of the sides of ABC intersect in the middle point M of BC and in the required centre O ; the angle bisectors of OM and BC will give the directions of the asymptotes. Alternatively, find D' the isogonal conjugate of D with respect to ABC ; if the diameter through D' of the circle ABC has ends ξ, η, then the asymptotes are the Simson Lines of ξ, η with respect to the triangle ABC. (Observe that, if D is the orthocentre of ABC, D' is the circumcentre and the points ξ, η are indeterminate ; if D is the circumcentre of ABC, ξ, η are the intersections of the circle ABC and the Euler Line and the rectangular hyperbola passes through the Symmedian point.)

Some elementary geometry will clear the air and a rough figure (not worth printing) will confirm the two following statements :

(i) If A' is the isogonal conjugate of A with respect to the triangle BCD, and B', C', D' are the isogonal conjugates of B, C, D with respect to the triangles CAD, ABD, ABC, then A', D' are reflexions in BC, etc., and therefore A is the centre of the circle $B'C'D'$, etc.

(ii) If P is any point on the rectangular hyperbola through A, B, C, D, denote by $\bar{P}$ the other end of the diameter through P of the rectangular hyperbola ; the familiar property that any chord of a rectangular hyperbola subtends equal or supplementary angles at P, $\bar{P}$ can be stated in the form " the chords PQ, $\bar{P}\bar{Q}$ of a rectangular hyperbola subtend equal or supplementary angles at any point of the rectangular hyperbola " and then it can be seen that the isogonal conjugate of P with respect to ABQ coincides with the isogonal conjugate of $\bar{Q}$ with respect to $AB\bar{P}$. It follows that the isogonal conjugates of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ with respect to BCD, CAD, ABD, ABC are the same point. Let us call this point K and consider some of its relations to the quadrangle A, B, C, D.

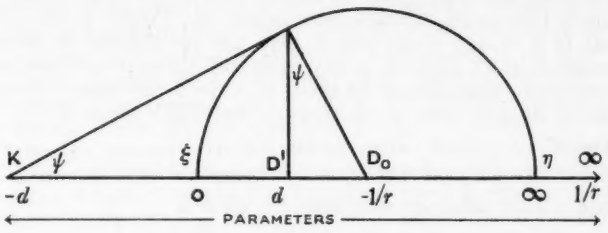
The coordinates of K are by § 2,

$$x=\frac{a+b+c+d}{1+abcd}, \quad y=\frac{1/a+1/b+1/c+1/d}{1+1/abcd},$$

if A, B, C, D are the points $(t, 1/t)$, for $t=a, b, c, d$; K and D' are inverse points with respect to the circle ABC, etc., and we shall first show that the pairs of tangents from K to the four circles BCD, CAD, ABD, ABC enclose the same angle 2ψ.

The figure shows six points on the associated diameter of the circle ABC (with centre D_0) and the parameters of their isogonal conjugates on the rectangular hyperbola ; if d is positive the sequence

of points requires $r(=abc)$ to be negative, so that three of the points A, B, C, D are on one branch of the rectangular hyperbola.



The cross-ratio of any four of these points is equal to that of the four corresponding parameters, and therefore

$$(D_0 \infty D'\eta) = (-1/r,\ 1/r,\ d,\ \infty),$$

giving
$$\frac{D_0D'}{D_0\eta} = \frac{d+1/r}{d-1/r}, \quad i.e.\ \sin\psi = \left|\frac{1+abcd}{1-abcd}\right|.$$

K is the common point of the associated diameters of the four circles BCD, etc., with centres A_0, etc., and is the centre of similitude of the quadrangles A', B', C', D' and A_0, B_0, C_0, D_0, for

$$\frac{KA'}{KA_0} = \frac{KB'}{KB_0} = \frac{KC'}{KC_0} = \frac{KD'}{KD_0} = \cos^2\psi.$$

5. *The isoptic point of a quadrangle.*

The relation $1 - \left(\frac{D_0K}{D_0\eta}\right)^2 = \frac{4abcd}{(1+abcd)^2}$ shows that, if K is outside one of the four circles, it is outside them all, and then the four circles when viewed from K will have the same angular magnitudes. Dr. G. T. Bennett, who was the first to discover its existence, called it the isoptic point ; he determined it as the common point of the six circles of similitude of the four circles BCD, etc., taken in pairs. It is easily identified as the centre of the circle through the reflexions of $\bar{D}$ in the sides of $ABCD$, and $\bar{D}$ can be determined without reference to the rectangular hyperbola as the common point of the circles $BC\alpha$, $CA\beta$, $AB\gamma$, $\alpha\beta\gamma$, where α, β, γ are the reflexions of D in the sides of ABC.

The isoptic point K has many properties which can be verified when its coordinates are known, but geometrical proofs may be elusive ; *e.g.* :

(i) If O is the centre of the rectangular hyperbola through A, B, C, D, then K is the Frégier point of O for the conic $ABCDO$; for the equation of this conic is

$$\gamma \equiv x^2 + \Sigma ab \,.\, xy + abcdy^2 - \Sigma a \,.\, x - \Sigma bcd \,.\, y = 0.$$

The same property in other words is " if O is a fixed point on a fixed conic, any rectangular hyperbola with its centre at O cuts the conic in four points, whose isoptic point is the Frégier point of O ".

(ii) If S is any conic (other than the rectangular hyperbola) through A, B, C, D and S' is any conic with its centre at O and with asymptotes perpendicular to those of S, then the intersections of S and S' lie on a circle whose centre is the middle point of OK.

For, if $$S \equiv \gamma + \kappa(xy - 1) = 0,$$

and $$S' \equiv abcd\, x^2 - (\kappa + \Sigma ab)xy + y^2 + \lambda = 0,$$

where κ, λ are arbitrary constants, the circle is

$$S + S' \equiv (1 + abcd)(x^2 + y^2) - \Sigma a \,.\, x - \Sigma bcd \,.\, y + \lambda - \kappa = 0.$$

If in particular we take S as the line pair BC, AD, and S' as the pair of perpendiculars from O to BC, AD, then K appears as the unique point with the property that the middle point of the joins of its reflexions in each pair of opposite sides of the quadrangle is the same point (O).

(iii) There is a unique parabola touching the tangents to the rectangular hyperbola at A, B, C, D; K is the pole of its directrix with respect to the rectangular hyperbola.

Finally there are two more very different properties communicated to me by Mr. J. H. Grace :

(iv) There is a unique circular cubic through the five points A, B, C, D, K having KA, KB, KC, KD as tangents at A, B, C, D.

(v) If in the Argand diagram the points A, B, C, D represent the complex variables $\alpha, \beta, \gamma, \delta$, *real* coefficients l, m, n, p can be found uniquely such that the equation

$$\frac{l}{z-\alpha} + \frac{m}{z-\beta} + \frac{n}{z-\gamma} + \frac{p}{z-\delta} = 0$$

is linear in z ; the solution of this linear equation is then represented by the isoptic point of A, B, C, D.

6. *Bibliographical note.*

The literature on the topics of isogonal conjugates, pedal circles and associated circles bearing the names of Brocard, Lemoine, Tucker, Neuberg, etc., etc., is immense; most of it appeared between 1870 and 1890 and numerous references will be found in Coolidge's *Treatise on the Circle and the Sphere* (Clarendon Press, 1916). Casey's *Sequel to Euclid* (Dublin University Press, 6th ed., 1892) is a mine of information, but the proofs are based entirely on the properties of lines and circles. A full bibliography up to 1895 was published by Vigarié in *Mathésis*, Series 2, Vol. vi, 1896.

P. W. W.

RECURRING DIGITS IN IRRATIONAL DECIMALS.

By R. L. Goodstein.

Is it always possible to determine a real number to an assigned number of decimal places? Can we always tell how many terms of a convergent sequence must be taken to determine its limit to, say, four decimal places? * How much information does the *convergence* of a sequence give us about the decimal expansion of the limit? These are the questions we shall consider; we shall find that the answer to the first two is "no". To the third question the answer is that convergence itself gives very little information but that "most" convergent sequences have in fact a subtler property which gives precise information about the expansion of the limit, the rate at which the successive digits of the expansion can be determined and the number of times a digit can recur. The questions we have raised are of *practical* importance for it is with the approximations-to-so-many-places, not with real numbers themselves, that all practical mathematics is concerned.

Let us consider the way in which the limit of a convergent sequence is *worked out*. Suppose we wish to determine to three decimal places the number "e" which is the limit of the sequence $s_n = \sum_{r=0}^{n} 1/r!$ The proof that s_n is convergent consists in showing that, for any n and p, s_{n+p} exceeds s_n by less than $1/n!$, so that in particular s_n differs from s_8 by less than $1/10^4$ for any $n \geqslant 8$. The evaluation of s_8 gives $s_8 = 2{\cdot}7182785\ldots$. Can we say that, to three places, $e = 2{\cdot}718$? We know that for any $n \geqslant 8$, $s_8 \leqslant s_n < s_8 + 1/10^4$ and so s_n lies between $2{\cdot}7182$ and $2{\cdot}71837\ldots$, that is to say, the digits 2718 are common to all s_n, $n \geqslant 8$, and this we may accept as the defining characteristic of the first four digits of the limit e. Unfortunately the situation is not always as simple as this. Let us imagine that for some convergent sequence t_n we have proved $t_{n+1} > t_n$, $t_{n+8} - t_8 < 1/10^4$ and $t_8 = 2{\cdot}2999993\ldots$. For $n \geqslant 8$, t_n lies between $2{\cdot}299$ and $2{\cdot}301$, but are the first four digits of the limit l of the sequence 2299 or 2300? The convergence condition fails to answer the question and we are obliged to work out t_9, t_{10}, t_{11}, and so on; if we find a t_n which commences with the digits 2300 then we know that these digits are common to all subsequent t_r, since t_n is increasing, but if we do not find such a t_n we cannot say whether the second digit of l is 2 or 3. There is no way of knowing in advance how many terms of the sequence we must work out to determine l even to one decimal place, nor can we know that l will be so determined however far we carry the calculation! It may be objected that we need only determine n_1 so that $t_n - t_{n_1} < 10^{-7}$ for $n \geqslant n_1$ (this n_1 is given of course by the proof of the conver-

* We are not interested, here, in the common distinction between fast and slowly convergent sequences, for even a fast convergent sequence may, as we shall see, yield the digits of its limit only with great reluctance.

gence of t_n) and then take the first two digits of t_{n_1}; but if $t_{n_1}=2{\cdot}29999996\ldots$ we are in no better a position. The next suggestion will probably be that in such a case l must equal 2·29 which is the same as 2·30; but what is the proof of this? No matter how many terms of the sequence we evaluate and find a recurrence of 9's some subsequent term may be greater than 2·3 (and so l necessarily greater than 2·3). A more fruitful line of argument is to observe that if l is proved to be irrational and if this proof fixes the number of times a digit may recur in l then we know for what value of n the recurrence of 9's in t_n must terminate and the digits of l are fixed—and if l is proved to be rational, and the proof furnishes us with a fraction p/q as the value of l then again the digits of l are known. Reflection reminds us, however, that the conventional proofs of irrationality are by *reductio ab absurdum* and supply us with no information about the recurrence of digits except that such recurrences must stop *somewhere*. Think, for instance, of Euclid's proof that $\sqrt{2}$ is irrational; does this tell us how often a 9 may recur in the expansion of $\sqrt{2}$? Moreover, we may find ourselves unable, as in the case of Euler's constant γ, either to prove that l is rational or to prove that it is irrational. To say that it must be one or the other does not give us the information we are seeking for. In any case, if we do not know the expansion of l, what is this *number* which must be rational or irrational? Without more information than convergence gives we shall have to admit that we may be unable to find the value of l to three places of decimals—and yet we apparently rely on convergence to give the expansions of the majority of real numbers, for the values of e^x, $\sin x$, $\log x$, etc., are all computed from convergent sequences and no tabulator has till now been obliged to omit from his tables a value of any of these functions.

The explanation of the tabulator's success is that the sequences he uses are (though he does not know it) not merely convergent but *strongly* convergent, a property defined as follows:

Denoting by $(p/q)_n$ the quotient to n places of p divided by q, a sequence of rationals s_n is said to be *strongly convergent* if we can determine a sequence n_k such that, for any k,

$$(s_m)_k=(s_{n_k})_k,\ m\geqslant n_k.$$

A strongly convergent sequence s_n determines the expansion of its limit digit by digit, for the first k digits of the limit are just those of $(s_{n_k})_k$.

It is obvious from the definition that a strongly convergent sequence is necessarily convergent. We shall examine what conditions added to convergence suffice to ensure strong convergence.

The first theorem we prove is that if a sequence s_n converges to a rational limit p/q (a fraction in its lowest terms) and if q contains a prime factor δ which is not a factor of 10, then s_n is strongly convergent. This result is not, itself, of much importance but

affords a simple introduction to the technique of the more important second theorem. The theorem depends upon the familiar fact that if q is a number of n digits then in the decimal expansion of p/q neither a 9 nor a 0 can recur more than $n-1$ times. For if $p/q = a_0 \cdot a_1 a_2 a_3 \dots a_r 00 \dots 0a_{r+\mu+1} \dots$, and if $a_0 \cdot a_1 a_2 \dots a_r = N/10^r$, then since q is divisible by δ and neither p nor 10^r is divisible by δ, $10^r p - qN$ is not divisible by δ and so not equal to 0, and therefore $|p/q - N/10^r| = |10^r p - qN|/q10^r \geqslant 1/q10^r \geqslant 1/10^{n+r}$. But

$$|p/q - N/10^r| = \cdot 00 \dots 0a_{r+\mu+1} \dots < 1/10^{r+\mu}$$

and so $r+\mu < n+r$, i.e. $\mu < n$.

Similarly if $p/q = a_0 \cdot a_1 a_2 \dots a_r 99 \dots 9a_{r+\mu+1} \dots$, then $\mu < n$.

Now the expansion of p/q is a recurring decimal since q contains a prime factor which is not a factor of 10, so we may suppose

$$p/q = a_0 \cdot a_1 a_2 \dots a_\kappa \dot{\alpha}_1 \alpha_2 \dots \dot{\alpha}_\nu.$$

Since s_n is convergent to p/q we can determine n_r such that

$$|p/q - s_m| < 1/10^{\nu+r} \text{ for any } m \geqslant n_r.$$

If $\nu = 1$, neither a 9 nor a 0 can occur in the decimal expansion of p/q after the κth place.

If $\nu > 1$ no digit in the expansion of p/q, after the κth place, can recur more than $\nu - 1$ times in succession; in particular neither a 9 nor a 0 can recur more than $\nu - 1$ times. Thus each s_m, for $m \geqslant n_r$, has the same integral part and first r decimal figures as p/q; i.e. $(s_m)_r = (p/q)_r$, $m \geqslant n_r$, and so s_n is strongly convergent.

Our second theorem is that if s_n is convergent but does not converge to a rational limit then s_n is strongly convergent. More precisely, s_n is strongly convergent if s_n is convergent and if we know two numeral functions $i(p, q)$ and $N(p, q)$ such that

$$|s_n - p/q| > 1/10^{i(p,q)}, \ n \geqslant N(p, q). \quad \dots\dots\dots\dots\dots\dots\text{A}$$

Since s_n is convergent we can find n_t such that, for $n \geqslant n_t$,

$$|s_n - s_{n_t}| < 1/10^{t+1}.$$

Let $s_{n_t} = \alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_t \alpha_{t+1} \dots$.

(1) If α_{t+1} is neither 0 nor 9,

then $s_n = \alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_t \beta_{t+1} \dots$ for any $n \geqslant n_t$.

(2) If $\alpha_{r+1} = \alpha_{r+2} = \dots = \alpha_{t+1} = 9, \ \alpha_r \neq 9,$

then $s_n = \alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_r 99 \dots 9\gamma_{t+2} \dots$

or $s_n = \alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_{r-1}(\alpha_r + 1) 00 \dots 0\delta_{t+2} \dots$.

(3) If $\alpha_{r+1} = \alpha_{r+2} = \dots = \alpha_{t+1} = 0, \ \alpha_r \neq 0,$

then $s_n = \alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_r 00 \dots 0\epsilon_{t+2} \dots$,

or $s_n = \alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_{r-1}(\alpha_r - 1) 99 \dots 9\theta_{t+2} \dots$.

Consider case (2). Write

$$\alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_{r-1}(\alpha_r + 1) = \nu/10^r \text{ and } i(\nu, 10^r) = I,\ N(\nu, 10^r) = J.$$

By virtue of condition A we can find a least T satisfying

$$t+1 < T \leqslant I$$

(the next step in the proof shows that $t+1$ is necessarily less than I), such that if $n \geqslant \max(n_t, J)$ then

$$|\, s_n - \nu/10^r \,| > 1/10^I,$$

and therefore if $s_n = \alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_r 99 \dots 9\gamma_{t+2}\gamma_{t+3} \dots$ we must have $\gamma_p = 9$ for $r < p < T$ and $\gamma_T \neq 9$ (of course T may equal $t+2$), and if

$$s_n = \alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_{r-1}(\alpha_r + 1)\,00 \dots 0\delta_{t+2}\delta_{t+3} \dots,$$

then $$\delta_p = 0 \text{ for } r < p < T \text{ and } \delta_T \neq 0.$$

But if $\mu = \max(n_I, J)$ then for $n \geqslant \mu$, $|\, s_n - s_\mu \,| < 1/10^I$, and so for $n \geqslant \mu$ *all* a_n are of the form

$$\alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_r 99 \dots 9\gamma_{t+2} \dots$$

or *all* s_n are of the form

$$\alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_{r-1}(\alpha_r + 1)\,00 \dots 0\delta_{t+2} \dots,$$

i.e. for $n \geqslant \mu$ all s_n have the same first t digits.

Similarly in case (3), if $\alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_r = \nu'/10^r$,

and if $$i(\nu', 10^r) = I',\ N(\nu', 10^r) = J',\ \mu' = \max(n_{I'}, J'),$$

then for $n \geqslant \mu'$ all s_n are of the form $\alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_r 00 \dots 0\epsilon_{t+2} \dots$ or all of the form $\alpha_0 \cdot \alpha_1 \alpha_2 \dots \alpha_{r-1}(\alpha_r - 1)\,99 \dots 9\theta_{t+2} \dots$, and therefore for $n \geqslant \mu'$ all s_n have the same first t digits.

Hence if $k = \max(\mu, \mu')$ we have, in every case, that all s_n, $n \geqslant k$, have the same first t digits and therefore s_n is strongly convergent.

By means of this theorem we readily deduce that the sequence $s_n = \sum_0^n 1/r!$, whose limit is e, is strongly convergent. We shall prove that $|\, s_n - p/q \,| > 1/10^{q!}$, and from this the required result follows immediately.

We have

$$|\, s_n - p/q \,| = \{1 + q + q(q-1) + \dots + q! - p(q-1)! + 1/(q+1) + \dots\}/q!.$$

But $q(q-1) + q(q-1)(q-2) + \dots + q!$ is divisible by $q-1$ (if $q > 1$) and $q+1 = q-1+2$ is not divisible by $q-1$ unless $q = 2$ or 3.

Leaving aside the cases $q = 1, 2, 3$, we see that since

$$1/(q+1) + 1/(q+1)(q+2) + \dots < 1/q,$$

therefore $$|\, s_n - p/q \,| \geqslant (1 - 1/q)/q! > 10^{-q!}.$$

For the cases $q = 1, 2, 3$, the difference $|\, s_n - p/q \,|$ is least when $p = 8$, $q = 3$ and $|\, s_n - 8/3 \,| > 1/24$.

Observe that we have shown that the decimal development of e differs from the expansion of p/q at the $(q!)$th digit at the latest, for

the first $q!$ digits of e are the same as those of s_n for a sufficiently great n, and s_n differs from p/q at the $(q!)$th place or earlier.

As another illustration of the application of Theorem 2 we shall prove that if a convergent sequence s_n is a root of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, with integral coefficients, i.e. if $f(s_n)$ converges to zero, then s_n is strongly convergent.

Since $f(p/q) = (a_n p^n + qI)/q^n$, where I is an integer, we see, if a_n is not divisible by q and if p/q is in its lowest terms, that

$$|a_n p^n + qI| \geqslant 1 \text{ and therefore } |f(p/q)| \geqslant 1/q^n.$$

Similarly $f(p/q) = (a_0 q^n + pI)/q^n$ and so, if a_0 is not divisible by q, $|f(p/q)| \geqslant 1/q^n$.

Thus to find if p/q is a root of $f(x)$ we need only consider values of p which divide a_0 and values of q which divide a_n. Therefore in a finite number of steps we can determine whether $f(x)$ has any rational roots. We shall suppose that s_n is a root of a polynomial with no rational roots.

$\{f(x) - f(y)\}/(x-y)$ is a continuous function of x and y and therefore bounded in a closed interval. Let M be the bound of $|\{f(x) - f(y)\}/(x-y)|$, then $|f(p/q) - f(s_m)| \leqslant |p/q - s_m| M$.

Since $f(s_m)$ converges to zero we can determine k so that

$$|f(s_m)| < 1/2q^n \text{ for } m \geqslant k.$$

But $|f(p/q)| \geqslant 1/q^n$, so that $|f(p/q) - f(s_m)| \geqslant 1/2q^n$

and therefore $|p/q - s_m| \geqslant 1/2Mq^n$, $m \geqslant k$,

whence the result stated follows from Theorem 2.

We conclude by showing that if x is a non-square number then a recurrence in the decimal expansion of $\sqrt{x}$ which commences at the nth digit will terminate before the $(2n)$th.

Let p/q be a fraction in its lowest terms and x a prime integer. Then $|x - p^2/q^2| = |xq^2 - p^2|/q^2 \geqslant 1/q^2$, for if p is divisible by x, then q is not, and so $q = mx + r$, $0 < r < x$, wherefore $xq^2 - p^2 = x(M(x) + r^2)$ which is not divisible by x^2, and so not zero, and if p is not divisible by x then $xq^2 - p^2$ is not divisible by x and so not zero.* Denoting by x_n the development of $\sqrt{x}$ to n decimal places, we can choose k so that for $n \geqslant k$,

$$x - x_n^2 < 1/2q^2$$

and so $|x_n^2 - p^2/q^2| \geqslant 1/2q^2$, whence $|x_n - p/q| \geqslant 1/2q^2(x_n + p/q)$; if $x_n \geqslant 2p/q$ then $|x_n - p/q| \geqslant p/q > 1/6pq$, and if $x_n < 2p/q$ then $x_n + p/q < 3p/q$ and so $|x_n - p/q| > 1/6pq$.

If p has k digits and q has l digits then $6pq < 10^{k+l+1}$ and so $|x_n - p/q| > 1/10^{k+l+1}$; therefore the expansions of $\sqrt{x}$ and p/q differ before the $(k+l+1)$th decimal place.

* It follows that the same inequality holds even when p, q are not relatively prime, for if $p = hp'$, $q = hq'$, where p', q' have no common factor, then

$$|xq^2 - p^2| = h^2 |xq'^2 - p'^2| \geqslant 1,$$

and so

$$|x - p^2/q^2| \geqslant 1/q^2.$$

Suppose that $\sqrt{x} = a_0 \cdot a_1 a_2 \ldots a_{n-1}\,\alpha\alpha \ldots \alpha\, a_{n+m+1} \ldots$.
Consider

$$a_0 \cdot a_1 a_2 \ldots a_{n-1}\dot{\alpha} = p/q = N/10^{n-1} + \alpha/9 \cdot 10^{n-1} = (9N+\alpha)/9 \cdot 10^{n-1}.$$

p has at least n digits, q at least $n-1$, and so the sum of the number of digits of p and the number of digits of q is $2n-1$. Therefore, since

$$x_t = a_0 \cdot a_1 a_2 \ldots a_{n-1}\,\alpha\alpha \ldots \alpha\, a_{n+m+1} \ldots a_t$$

provided $t > n+m+1$, it follows that

$$| a_0 \cdot a_1 a_2 \ldots a_t - a_0 \cdot a_1 a_2 \ldots a_{n-1}\dot{\alpha} | > 1/10^{2n}.$$

But $| a_0 \cdot a_1 a_2 \ldots a_t - a_0 \cdot a_1 a_2 \ldots a_{n-1}\dot{\alpha} | < 1/10^{n+m}$ so that $m < n$, i.e. a recurrence commencing at the nth digit terminates before the $(2n)$th.

The method of this discussion is quite general and may be used to delimit the number of recurrences of a digit in the decimal expansion of the limit of any convergent sequence s_n which satisfies an inequality of the form

$$| s_n - p/q | > 1/10^{i(p,q)} \text{ for } n \geqslant \text{some } N(p, q)$$

(and which is therefore strongly convergent). R. L. G.

ERRATA.

Vol. XXV, No. 263, February 1941.

" Poncelet's Poristic Polygons ", by F. H. V. Gulasekharam.

P. 32, l. 2. *For* $N_{2m}N'_{2m}$ *read* $N_mN'_m$.
l. 4. *For* N_{2m} *read* N_m.
l. 6. *For* the sign of d *read* the sign of d when m is odd, and of R when m is even.

P. 34, ll. 29, 35, 39. *For* (9·1) *read* (10·1).
l. 41. *For* (9·4) *read* (10·4).

P. 35, l. 23. *For* V^6, V^7 *read* V_6, V_7. F. H. V. G.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

THE ELLIPTICAL ORBIT OF THE EARTH.

By Sir Percy Nunn.

Preliminary note. Beginners find two ways of regarding an ellipse easy to understand. The first presents it as the curve traced out by a point P (Fig. 1) when $r+r'$, the sum of P's distances from two fixed points F and F', is a constant, $2a$. From this definition come the equivalences $AA'=2a$, $CA=CA'=a$, and $CF=CF'=ae$, where e is a fraction less than unity. C is the " centre " of the ellipse, F and F' its " foci ", $CA=a$ the " semi-axis major ", $CB = b$ the " semi-axis minor ", and e the " eccentricity ".

The second way of obtaining the curve is to reduce (or extend) the ordinates of a circle in a constant ratio. To present this idea vividly one may imagine vertical rain falling upon a circular disc held parallel to a horizontal surface. The dry patch beneath the disc is, of course, circular, but it becomes elliptical when the disc is tilted.

To prove that these processes result in identical curves we start with Fig. 1 again, without the fixed points F and F', but with a circle of

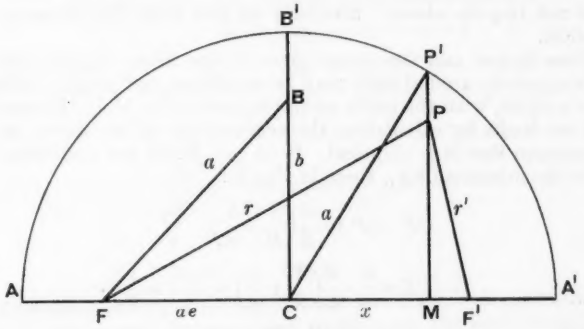


Fig. 1.

radius a drawn upon AA' as diameter. $B'C$ is the ordinate through C. Let the ratio of reduction, arbitrarily chosen, be b/a ; the then reduced ordinate at C becomes $BC=b$. Suppose a circle to be drawn with B as centre and a as radius, and let it cut AA' in the two points F and F'. Also let $CF=CF'=ae$. Then from $\triangle BCF$ we have

$$a^2=(ae)^2+b^2 \text{ or } b^2=a^2(1-e^2).$$

Let $P'M$ be any other ordinate of the circle, and let it be reduced to $PM=(b/a)P'M$. Also let the distances PF and PF' be r and r'. Then the triangles PMF and $P'MC$ give us :

$$r^2=(ae+x)^2+(b/a)^2(a^2-x^2)=(ae+x)^2+(1-e^2)(a^2-x^2)=(a+ex)^2.$$

Hence $FP = r = a + ex$. Similarly we can show that $r' = a - ex$; so that $r + r' = 2a$. It follows that the curve given by the second process is an ellipse as defined by the first process, F and F' being the foci and e the eccentricity.

The foregoing proof is set out here because I am not aware that it has been given elsewhere. The partial result $r = a + ex$ is needed in what follows.

The Sun's apparent orbit.

The earlier astronomers believed that the sun travels round the earth annually in a circle; but this cannot be true, since modern measurements show that its disc varies in apparent diameter during the year.* For instance, the *Nautical Almanac* for 1938 states that the sun's semi-diameter (S.D.) was least about mid-day on 3rd July, when it measured 945·36″, and was greatest about the beginning of 3rd January, when it was 977·52″. That is, in England the sun is nearest the earth in winter and farthest off in summer. In the southern Dominions the opposite relation obtains. The sun is so remote that the figures quoted may be taken to be inversely proportional to its distance from the earth. Thus on 3rd January the distance was $k/977{\cdot}52$, where k is a constant whose value we need not trouble about. Similarly on 3rd July, the distance was $k/945{\cdot}36$.

These figures and the others given in the table suggest that the sun's apparent annual path may be an ellipse, not greatly differing from a circle, with the earth occupying one of its foci. To test this idea we begin by calculating the eccentricity of the curve, on the assumption that it is elliptical. If we put M for the maximum and m for the minimum S.D., then, in Fig. 2:

$$CP = \tfrac{1}{2}PA = \frac{k}{2}\left(\frac{1}{M} + \frac{1}{m}\right)$$

and

$$CE = \frac{k}{m} - \frac{k}{2}\left(\frac{1}{M} + \frac{1}{m}\right) = \frac{k}{2}\left(\frac{1}{m} - \frac{1}{M}\right).$$

Thus

$$e = CE/CP = (M - m)/(M + m)$$

Substituting the values of M and m, we find:

$$e = 32{\cdot}16/1922{\cdot}88 = 0{\cdot}016724.$$

The next step is to find a formula connecting the sun's semi-diameter with its position in the hypothetical ellipse. In the figure we have:

$$r = a + ex \text{ (as above)} \quad \text{and} \quad r\cos\theta = ae + x,$$

$$i.e.\ ex = r - a \quad \text{and} \quad ex = er\cos\theta - ae^2,$$

* At the beginning of the seventeenth century it was much easier to observe that the sun's movement along its apparent annual path varies in speed. The theory of the elliptical orbit started from such observations, but those assumed in this article are vastly easier to treat mathematically.

whence $$r(1-e\cos\theta)=a(1-e^2)$$

and $$\frac{k}{S}(1-e\cos\theta)=a(1-e^2), \quad \text{.....(i)}$$

S being the sun's semi-diameter. If S_0 is the S.D. when $\theta = 90°$, this equation reduces to

$$k/S_0=a(1-e^2). \quad \text{.....(ii)}$$

The combination of (i) and (ii) gives us the formula we require, namely :

$$S=S_0(1-e\cos\theta). \quad \text{.....(iii)}$$

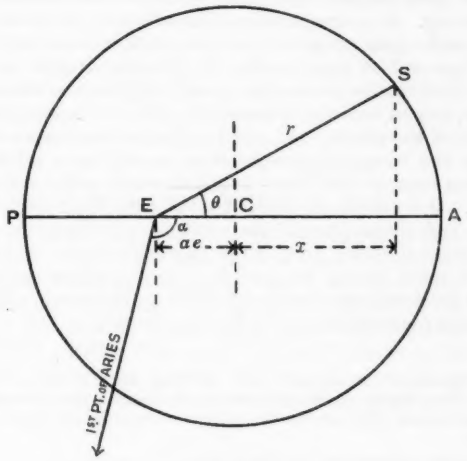


FIG. 2.

Now the *Nautical Almanac* does not give a table of the values of θ, but it does record, for every day in the year, the sun's longitude—that is, the angle through which it has moved along its orbit from the "first point of Aries", where that orbit crosses the celestial equator on or about 21st March. In 1938, when the sun reached A (in Fig. 2) at mid-day on 3rd July, its longitude (marked α in the figure) was 101°. Hence the longitude when $\theta = 90°$ was 191°. The table indicates that this happened on 5th October, and that the sun's S.D. was then 961·2″. Thus (iii) takes the final form :

$$S=961{\cdot}2''(1-0{\cdot}016724\cos\theta). \quad \text{.....(iv)}$$

The simple case of $\theta = 60°$ may be used to test its validity. We have :

$$S=961{\cdot}2''\times 0{\cdot}991638$$
$$=953{\cdot}16''.$$

This agrees with the S.D. recorded * in the *Almanac* for 4th September, the day on which the sun's longitude is $60° + 101° = 161°$. Similar tests, for dates chosen at random, will leave no doubt that the sun's orbit is, as we guessed, an ellipse of small eccentricity, one of whose foci is occupied by the earth.

The assumption, implicit above, that we are concerned with movement of the sun about the earth, not of the earth about the sun, has probably ruffled the feelings of many readers. Yet (as history shows) it is consonant with common-sense. From day to day throughout the year the sun shines forth from a different spot in the background of fixed stars ; and if we may not say that it *moves* across that background, the word " moves " seems to be deprived of its meaning. It is true that a solar observer, if there were one, could say with equal propriety that the *earth* was moving across the same background of fixed stars. Copernicus taught astronomers to prefer the heliocentric to the geocentric point of view because the former brings order and simplicity into our conception of the movements of the planets as a whole. But to insist upon that point of view in the foregoing investigation would have made an easy problem too complicated for a beginner—and unnaturally complicated, too. It is enough to realise that if, from the geocentric standpoint, the sun moves round the earth in an ellipse in which the earth occupies a focus, then, from the heliocentric standpoint the earth must move round the sun in an equal ellipse with the sun occupying the focus previously *not* filled by the earth. The way to the Copernican generalisation is thus made clear.

T. P. N.

* Strictly speaking, the entries in the *Nautical Almanac* are not records but predictions. But, being based upon records of observations and amply verified by later observations, they may for our purpose be regarded as records of measurements made.

GLEANINGS FAR AND NEAR.

1375. WHICH?

" Observer ", of London, W.C., writes : A friend of mine says that if two bombs, one weighing 500 lb. and the other 100 lb., were dropped from a plane flying at 10,000 ft., at exactly the same time, the heavier bomb would hit the ground first. I say both would drop together. Who's right?

Answer : If a feather and a pebble were dropped from a height of 10 ft., would the pebble float down gently with the feather?—*Daily Mirror*, Sept. 6, 1940. [Per Mr. G. H. Grattan Guinness.]

1376. But here something must first be said about equilibrium. It is known that when two things act against each other, and when one reacts and resists as much as the other acts and impels, neither of them has any force, because on both sides there is the same power, and in such case both may be acted upon by a third at pleasure ; for when two things from equal opposition have no force, the force of a third does all, as if there were no opposition. Such is the equilibrium between Heaven and Hell.—E. Swedenborg, *Heaven and Hell*, § 537. [Per Mr. R. F. Cyster.]

AL CAPONE AND THE DEATH RAY.

By R. C. Lyness.

Problems about the American city road-system with its streets running east and west and its avenues running north and south have appeared from time to time in examination papers and are interesting because they give relations between binomial coefficients which are harder to get by straightforward algebra. After illustrating some of the methods by which difficult-looking results can be obtained simply, I shall obtain a simple-looking result with difficulty and ask for an easier way of getting it.

The streets and avenues are numbered consecutively; the more northerly the street, the more easterly the avenue, the higher its number. Every two neighbouring streets and neighbouring avenues are the same distance apart. The intersection of the pth avenue with the qth street is defined to be the *corner* (p, q). Any route by road from (p, q) to (r, s) which proceeds always in a northerly or easterly direction is called a *path* from (p, q) to (r, s). We cannot have a path from (p, q) to (r, s) unless $r \geqslant p$, $s \geqslant q$.

The number of paths from $(0, 0)$ to (p, q) is $\binom{p+q}{q}$, for the traveller must leave $(p+q)$ corners in his journey, q and only q of which he must leave in a northerly direction. There is a one-one correspondence between every path and a selection of q from the first $p+q$ natural numbers. For instance, the path from $(0, 0)$ to $(3, 3)$ which corresponds to the selection of the 3rd, 5th and 6th corners looks like a wide W rotated through 45° about its bottom left-hand point. It follows directly that the number of paths from (p, q) to (m, n), being the same as the number of paths from $(0, 0)$ to $(m-p, n-q)$, is

$$\binom{\overline{m+n} - \overline{p+q}}{n-q}.$$

The number of paths from $(0, 0)$ to (m, n) which pass through (p, q) is equal to the number of paths from $(0, 0)$ to (p, q) multiplied by the number of paths from (p, q) to (m, n) and is hence

$$\binom{p+q}{q}\binom{\overline{m+n} - \overline{p+q}}{n-q}.$$

So

$$\binom{p+q}{q}\binom{\overline{m+n} - \overline{p+q}}{n-q} \Big/ \binom{m+n}{n}$$

is the probability that Al Capone in travelling from $(0, 0)$ to (m, n) passes over a particular corner (p, q) where the rival gang has a bomb waiting for him.

A *barrier* is a straight line across which every path between two points must pass. If we draw a barrier from (a, q) to $(a+q, 0)$, where $a \geqslant 0$, $p \geqslant a+q$ and assert the fact that the number of paths

from $(0, 0)$ to (p, q) is equal to the sum of the numbers of paths from $(0, 0)$ to (p, q) which pass through $(a+r, q-r)$, $r=0, 1, \ldots q$, we get Van der Monde's theorem, namely,

$$\binom{p+q}{q} = \sum_{r=0}^{q} \binom{a+q}{q-r}\binom{p-a}{r}.$$

The restrictions on p and a can now be removed by the usual argument; thus, both sides are polynomials of the qth degree in p and are equal for more than q values of p.

Next take a line just east of mth avenue as a barrier and consider paths from $(0, 0)$ to $(m+n+1-q, q)$. The number of such paths which cross the barrier just east of (m, r) is equal to the number of paths from $(0, 0)$ to (m, r) multiplied by the number from $(m+1, r)$ to $(m+n+1-q, q)$. Summing from $r=0$ to $r=q$, we have

$$\binom{m+n+1}{q} = \sum_{r=0}^{q} \binom{m+r}{r}\binom{n-r}{q-r}.$$

This is only Van der Monde in disguise, as we can see at once by making the substitution

$$\binom{l}{t} = (-)^t \binom{t-l-1}{t}.$$

We get more complicated results by taking other barriers with negative gradients. For instance, a barrier of gradient $-b$ passing through $(a, 0)$ with paths from $(0, 0)$ to $(m-q+n, q)$ gives the identity,

$$\begin{aligned}\binom{a}{0}\binom{m}{q} &+ \sum_{1}^{b-1}\binom{a-1+r}{r}\binom{m-r}{q-r} + \binom{a-1+b}{b}\binom{m+1-b}{q-b} \\ &+ \sum_{b+1}^{2b-1}\binom{a-2+r}{r}\binom{m+1-r}{q-r} + \binom{a-2+2b}{2b}\binom{m+2-2b}{q-2b} \\ &+ \sum_{2b+1}^{3b-1}\binom{a-3+r}{r}\binom{m+2-r}{q-r} + \ldots = \binom{m+a}{q}.\end{aligned}$$

Barriers with positive gradients have the peculiarity that a path may cross the barrier more than once, and it is with them that the main problem of this article is concerned

Suppose that a barrier connects the points $(r\overline{\alpha-1}+\epsilon, r)$ $r=0, 1 \ldots q$, where α is a positive integer and ϵ is positive and very small. The problem is to find an expression for u_n, the number of paths from $(0, 0)$ to $(n\overline{\alpha-1}, n)$ which do not cross the barrier $(r\overline{\alpha-1}+\epsilon, r)$. All paths from $(0, 0)$ to $(q\overline{\alpha-1}+t+1, q)$, where $t>-1$, must cross the barrier at least once. The number of such paths which first cross the barrier at $(r\overline{\alpha-1}+\epsilon, r)$ is u_r multiplied by the number of paths from $(r\overline{\alpha-1}+1, r)$ to $(q\overline{\alpha-1}+t+1, q)$. Hence summing from 0 to q,

$$\binom{q\alpha+t+1}{q} = \sum_{r=0}^{q} u_r \binom{\overline{q-r}\alpha+t}{q-r}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Since u_r is independent of t, this result holds for all t.

Consider now the series

$$y = u_0 + u_1 x + u_2 x^2 + \ldots + u_r x^r + \ldots ;$$

since $0 < u_r < \binom{r\alpha}{r}$ the series is absolutely convergent for sufficiently small values of x and so also is the series

$$f(t) = \binom{t}{0} + \binom{\alpha + t}{1} x + \ldots + \binom{s\alpha + t}{s} x^s + \ldots .$$

The coefficient of x^s in the product $y \,.\, f(t)$ is

$$\sum_{r=0}^{s} u_r \binom{\overline{s-r}\alpha + t}{s-r},$$

which, by (i) $$= \binom{s\alpha + t + 1}{s}.$$

Thus $f(t) \,.\, y = f(t+1)$,

and $f(t) \,.\, y^m = f(t+1) \,.\, y^{m-1} = f(t+2) \,.\, y^{m-2} = \ldots = f(t+m)$.

In particular,

$f(t) \,.\, y^\alpha = f(t+\alpha)$.

$$f(t) \,.\, xy^\alpha = \binom{t+\alpha}{0} x + \binom{t+2\alpha}{1} x^2 + \ldots + \binom{t+r\alpha}{r-1} x^r + \ldots ,$$

$$f(t) = \binom{t}{0} + \binom{t+\alpha}{1} x + \binom{t+2\alpha}{2} x^2 + \ldots + \binom{t+r\alpha}{r} x^r + \ldots .$$

Adding, $f(t) \,.\, (1 + xy^\alpha)$

$$= \binom{t}{0} + \binom{t+1+\alpha}{1} x + \binom{t+1+2\alpha}{2} x^2 + \ldots + \binom{t+1+r\alpha}{r} x^r + \ldots ,$$

$$= f(t+1)$$

$$= f(t) \,.\, y.$$

Hence $$y = 1 + xy^\alpha.$$

Differentiating, $$y_1 (1 - x\alpha y^{\alpha-1}) = y^\alpha ;$$

multiplying by $f(0)$, $y_1 \{f(0) - x\alpha f(\alpha - 1)\} = f(\alpha)$.

But

$$x\alpha f(\alpha - 1) = \alpha \binom{\alpha - 1}{0} x + \alpha \binom{2\alpha - 1}{1} x^2 + \ldots + \alpha \binom{\overline{r+1}\alpha - 1}{r} x^r + \ldots$$

$$= \binom{\alpha}{1} x + \binom{2\alpha}{2} x^2 + \ldots + \binom{\overline{r+1}\alpha}{r+1} x^r + \ldots$$

$$= f(0) - 1.$$

Hence $$y_1 = f(\alpha),$$

that is, $$\sum_1^\infty r u_r x^{r-1} = \sum_1^\infty \binom{r\alpha}{r-1} x^{r-1},$$

from which $$u_n = \frac{1}{n}\binom{n\alpha}{n-1}, \quad n>0,$$

or $$u_n = \binom{n\alpha}{n} \Big/ (n\alpha - n + 1).$$

Now the number of paths from (0, 0) to $(n\overline{\alpha-1}, n)$ is $\binom{n\alpha}{n}$.

Hence the probability that any such path does not cut the barrier $(r\overline{\alpha-1}+\epsilon, r)$ is $1/\{n(\alpha-1)+1\}$.

Putting this in another way, if m and n are positive integers, the number of paths from (0, 0) to (n, mn) is $(mn+1)$ times the number of such paths which keep to one of the sides of the line joining (0, 0) to (n, mn). Surely such a simple result can be more simply proved.

By the same methods it is easy to show that if the barrier is the line joining the points $(b+r\overline{\alpha-1}, r)$ and v_n is the number of paths from (0, 0) to $(b+n\overline{\alpha-1}, n)$ which do not cross the barrier, then

$$\binom{q\alpha+t+b+1}{q} = \sum_0^q v_r \binom{\overline{q-r\alpha}+t}{q-r}, \quad \text{.....(ii)}$$

and if $$z = \sum_0^\infty v_r x^r,$$

then $$z \cdot f(t) = f(t+b+1)$$
$$= y^{b+1} f(t),$$

so that $$z = y^{b+1}$$

and $$z_1 = (b+1) y^b y_1 = (b+1) f(\alpha+b).$$

Thus $$v_n = \frac{b+1}{n} \cdot \binom{n\alpha+b}{n-1}, \quad n>0$$

or $$v_n = \frac{b+1}{n(\alpha-1)+b+1} \binom{n\alpha+b}{n}.$$

Rewriting this result too, if m, n, c are positive integers, then the number of paths from (0, 0) to the point P $(n, mn+c)$ is

$$(mn+c+1)/(c+1)$$

times the number of such paths which do not cross the line of gradient m through P.

The application of this to Al Capone and the rival gang is obvious. Capone sets out by armoured car from (0, 0) for the rival gang's headquarters at $(n, mn+c)$. Unknown to him the rival gang has a death ray sent out from its headquarters in a direction $S.\ \text{arc cot}\ m.W.$

Then Al's chance of getting to his enemy alive is

$$(c+1)/(mn+c+1).$$

Note that incidentally we have shown that if $y=1+xy^{\alpha}$ where x and α are given, then one value of y is such that

$$y^{\beta}=1+\beta\sum_{1}^{\infty}\binom{r\alpha+\beta-1}{r-1}\frac{x^r}{r},$$

where α and β are positive integers and x is sufficiently small (actually $|x|<(\alpha-1)^{\alpha-1}/\alpha^{\alpha}$).

Also writing $t=m-q\alpha$ and $\beta=b+1$ in (ii), we have

$$\binom{m+\beta}{q}-\binom{m}{q}=\beta\sum_{1}^{q}\frac{1}{r}\binom{r\alpha+\beta-1}{r-1}\binom{m-r\alpha}{q-r},$$

an identity which holds for all m, β, α and positive integral q.

R. C. L.

1377. He (the Archbishop) . . . continued to read mathematics for pleasure for some years after he left Oxford, and mastered some part at least of Sir W. R. Hamilton's very difficult work on Quaternions. . . . His favourite subject (at Rugby) was geometrical conic sections; and we often compared solutions of the more difficult problems that came out in new text-books. Mine were often shorter, but seemed to him, and really were, accidental relations. His were almost invariably deduced from elementary properties of the figure, and were therefore far better teaching. (p. 59.)

1378. I (Archdeacon Wilson) was teaching geometry of three dimensions at Rugby, and I showed to Dr. Temple a boy's solution of the problem, " How many spheres can be described to touch the four planes, all indefinitely produced, which form the surface of an irregular tetrahedron? " The boy found, of course, that one could be inscribed, and four escribed to the four faces, touching them externally, and added that six more could be escribed to what might be called the " hip-roofed " spaces of which each edge formed the ridge. Dr. Temple *instantly* saw that if a sphere could be escribed in one of these " hip-roofed " spaces, it could not be escribed in the space determined by the opposite edge of the tetrahedron, the " hips " being necessarily inclined at a wider angle than the " ridge slopes " in one of each pair. Readers can test their own power of such visualisation. (p. 59.)

1379. I (Archdeacon Wilson) sent the problem—(" To give a proof of the fact that four colours were sufficient for the colouring of any map, say of counties or other areas, however interlaced, so that no two areas which had a boundary—not a point—in common should have the same colour ")—to Dr. Temple when Bishop of London. Months afterwards he wrote to me in May 1889 : " I wrote the enclosed at an evening meeting of the —— Society, when the vehement Mr. —— was saying what I did not care to listen to." . . . The story is incomplete without its sequel. The morning papers reported that " The Bishop of London was greatly interested in Mr. ——'s speech, and was observed to be taking notes all the time he was speaking." (p. 60.)

1380. I (Archdeacon Wilson) once challenged him to Kirkman's well-known problem of the fifteen young ladies. . . . He brought me the solution in forty minutes. (p. 61.)

Gleanings 1377-1380 are from *Memoirs of Archbishop Frederick Temple*, I. [Per Mr. A. F. Mackenzie.]

THE ORTHOPOLAR CIRCLE.

By F. H. V. Gulasekharam.

1. *Definition :* ABC is a triangle ; $A'B'C'$ is its medial triangle ; L, M, N are the feet of the perpendiculars from A, B, C respectively on a straight line ; $\sigma_1, \sigma_2, \sigma_3$ are three circles having their centres at A', B', C' respectively such that σ_1 passes through M and N, σ_2 through N and L, and σ_3 through L and M. Then it is readily seen that the radical axes of the three circles taken in pairs meet in a point W—the radical centre of the circles. Again, since L is a common point of σ_2 and σ_3, their radical axis is the line through L at right angles to their line of centres $B'C'$, and hence to BC. Hence the lines through L, M, N at right angles to BC, CA, AB respectively meet at W, which is said to be the orthopole of the line LMN. The common radical circle of $\sigma_1, \sigma_2, \sigma_3$ is called the *Orthopolar* circle of the line LMN. Let us consider a few applications of the properties of this circle and its centre.

2. *The locus-equation of the joint-director circle of the confocal conics* whose *areal envelope* equations are

$$\phi \equiv ul^2 + vm^2 + wn^2 + 2fmn + 2gnl + 2hlm = 0,$$

and

$$\phi - \lambda\Omega = 0,$$

where

$$\Omega \equiv \Sigma a^2l^2 - 2\Sigma bcmn \cos A,$$

is

$$uD_{AA} + vD_{BB} + wD_{CC} + 2fD_{BC} + 2gD_{CA} + 2hD_{AB} - 4\lambda M^2(x+y+z)^2 = 0,$$

where D_{AA} = the power of $P(x, y, z)$ with respect to the point-circle at A, and therefore equal to $(c^2y + b^2z)(\Sigma x) - \Sigma a^2yz$; D_{BC} = the power of P with respect to the circle on BC as diameter, and therefore equal to $(2Mx \cot A)(\Sigma x) - \Sigma a^2yz$; and so on ; and M is the area of the triangle of reference. (See Note 1449, *Gazette* No. 259).

If the two confocals are central conics, their director circles and the joint-director circle are concentric with the conics ; if ρ_1, ρ_2, ρ_0 respectively be the radii of these circles, then $2\rho_0^2 = \rho_1^2 + \rho_2^2$.

If the confocals are parabolas, the joint-orthoptic locus is a straight line parallel to their directrices and *mid-way* between them.

The last two results are well known.

3. *The auxiliary circle of a conic.*

Of the system of conics confocal with a conic whose foci are (S, S'), one is the point-pair (S, S'). The corresponding joint-director circle is the locus of intersection of a variable tangent to the conic and the perpendicular to it from either focus ; it is therefore the auxiliary circle of the conic.

The pedal circle of the isogonal pair (P, P') with respect to a triangle ABC is the auxiliary circle of the inconic of $\triangle ABC$ whose foci are (P, P').

If $P \equiv (x_1, y_1, z_1)$ with respect to $\triangle ABC$, then the envelope equation of the foci (P, P') is

$$(x_1l + y_1m + z_1n)(a^2l/x_1 + b^2m/y_1 + c^2n/z_1) = 0,$$

and that of the inconic is

$$\Sigma(b^2z_1/y_1 + c^2y_1/z_1 + 2bc\cos A)\, mn = 0.$$

The equation of the *pedal circle* can therefore be written in the form

$$\Sigma(b^2z_1/y_1 + c^2y_1/z_1 + 2bc\cos A)\, D_{BC} + 4M^2(x+y+z)^2 = 0.$$

4. *Ramaswami Aiyar's Theorem.*

The orthopolar circle of a straight line with respect to a triangle cuts orthogonally the joint-director circle of any inconic X of the triangle and the confocal conic Y which touches the given line.

[Reference has already been made to this useful theorem on page 258 of *Math. Gazette* No. 210.]

The following proof which I deduced from my result quoted in § 2 should be of interest :

Taking the given triangle ABC for triangle of reference, let p, q, r be the perpendicular distances AL, BM, CN of the given line from A, B, C. Then it is well known that

(i) $\Omega(p, q, r) = 4M^2$,

(ii) $px + qy + rz = 0$ is the *areal* equation of the given line.

Now the envelope equation of *any* inconic of the triangle ABC is

$$\phi \equiv fmn + gnl + hlm = 0,$$

and that of a confocal is

$$\phi - \lambda\Omega = 0.$$

The equation of the joint-director circle of these two confocals is

$$fD_{BC} + gD_{CA} + hD_{AB} - 4\lambda M^2(x+y+z)^2 = 0.$$

If the conic $\phi - \lambda\Omega = 0$ touches the line (p, q, r), then $4\lambda M^2 = \Sigma fqr$, and the equation of the corresponding joint-director circle can then be written in the form

$$fS_1 + gS_2 + hS_3 = 0,$$

where

$$S_1 \equiv D_{BC} - qr(x+y+z)^2,$$

$$S_2 \equiv D_{CA} - rp(x+y+z)^2,$$

$$S_3 \equiv D_{AB} - pq(x+y+z)^2.$$

Now the circles $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ whose centres are at A', B', C' have a unique common radical circle, and this cuts orthogonally the circle $fS_1 + gS_2 + hS_3 = 0$, for *all values* of f, g, h.

If we examine the equation $S_1 = 0$, we readily see that it represents the joint-director circle of the confocals $mn = 0$ and $4M^2mn - qr\Omega = 0$. Hence $S_1 = 0$ is the auxiliary circle of the conic which has its foci at B, C and touches the line (p, q, r), and is, therefore, identical with the circle σ_1 of § 1; similarly the circles $S_2 = 0$ and $S_3 = 0$ are identical with σ_2 and σ_3 respectively. Hence the joint-director circle of the inconic $\phi = 0$ and the confocal which

touches the line (p, q, r) is cut orthogonally by the common radical circle of σ_1, σ_2, σ_3, *i.e.* by the orthopolar circle * of the line (p, q, r).

5. *Deductions from Aiyar's Theorem.*

(1) *The orthopolar circle of a line cuts orthogonally the pedal circle of every point on the line.*

For, the pedal circle of any point P on the line is the joint-director circle of the inconic which has P and its isogonal conjugate P' for foci and the confocal (*i.e.* the point-pair P, P') which touches the line.

(2) *The orthopole of a line is the intersection of the Simson's (pedal) lines of the points in which the line cuts the circum-circle of the triangle.*

For, from (1) above, the pedal lines of the two points cut orthogonally the orthopolar circle of the given line, and therefore intersect at the centre of the circle.

(3) *The orthopolar circle of a diameter of the circum-circle of the triangle is a point-circle whose centre lies on the* N.P. *circle of the triangle, and is the common point of the pedal circles of points on the diameter.*

For, the pedal circle of every point on the diameter (the N.P. circle being one of them) cuts orthogonally the orthopolar circle of the diameter, while the orthopole itself is a point on the N.P. circle, being the point of intersection of the pedal lines at the extremities of the diameter.

6. *The Theorem of Note* 1281 (*Gazette*, No. 248).

The orthopole of any one of four straight lines with respect to the triangle formed by the other three lies on the line of orthocentres of the four triangles formed by the four lines.

This follows from 5 (1) above. For, let a, b, c, d be the four lines, and let the points bc, ca, ab, ad, bd, cd be denoted by A, B, C, A_1, B_1, C_1 respectively.

Now the orthopolar circle of d with respect to the triangle abc cuts orthogonally the pedal circles (with respect to the same triangle) of the points A_1, B_1, C_1 which lie on d. These pedal circles are the circles on AA_1, BB_1, CC_1 as diameters, and they are known to be members of the coaxal system of director circles of conics which touch the lines a, b, c, d. *The radical axis of the system is the directrix of the unique parabola which touches a, b, c, d and therefore contains the orthocentres of the four triangles formed by the lines.* Hence the orthopole of d with respect to $\triangle abc$ lies on the line of orthocentres of the four triangles; similarly for the orthopole of any line with respect to the triangle formed by the other three lines.

* The equation of this circle can be shown to be

$$\Sigma bcx^2 \cos A + [\Sigma(-qr + rp + pq)x](\Sigma x) = 0,$$

7. *The centre and radius of the orthopolar circle of the line* (p, q, r).

If $W \equiv (x, y, z)$, where $x+y+z=1$, be the orthopole, and ρ be the radius of the orthopolar circle, then since the circle is the common radical circle of the circles $S_1=0$, $S_2=0$, $S_3=0$ of § 4, we have

$$\rho^2 = D_{BC} - qr = D_{CA} - rp = D_{AB} - pq. \quad \ldots\ldots\ldots\ldots\ldots (7{\cdot}1)$$

If we put $k=\rho^2+\Sigma a^2yz$, we get

$$2Mx = (k+qr)\tan A,$$
$$2My = (k+rp)\tan B,$$
$$2Mz = (k+pq)\tan C;$$

so that

$$k = (2M - \Sigma qr \tan A)\cot A \cot B \cot C. \quad \ldots\ldots\ldots\ldots (7{\cdot}2)$$

Hence

$$2Mx = qr\tan A + (2M - \Sigma qr \tan A)\cot B \cot C, \quad \ldots\ldots (7{\cdot}3)$$

with similar expressions for $2My$ and $2Mz$.

Now when $x=0$, that is when the orthopole of the line (p, q, r) lies on BC, the line must either be at right angles to BC or pass through the other extremity of the diameter through A of the circle ABC. Hence if we write

$$\xi_1 = -ap + bq\cos C + cr\cos B,$$
$$\xi_2 = -bq + cr\cos A + ap\cos C,$$
$$\xi_3 = -cr + ap\cos B + bq\cos A,$$
$$\eta_1 = -ap\cos B\cos C + bq\cos B + cr\cos C,$$

and so on, we see that when $x=0$, p, q, r satisfy the equation $\xi_1\eta_1=0$. Hence from (7·3), using the identity (i) of § 4, the equations for x, y, z can be written in the form

$$x/(\xi_1\eta_1 \sin A) = y/(\xi_2\eta_2 \sin B)$$
$$= z/(\xi_3\eta_3 \sin C)$$
$$= 1/(4M^2 \sin A \sin B \sin C). \quad \ldots\ldots\ldots (7{\cdot}4)$$

Again $\rho^2 = k - \Sigma a^2yz.$

Hence from (7·2) and (7·3) we can express ρ^2 in terms of p, q, r.

Now, we know that $\rho=0$ when the line (p, q, r) is at right angles to BC, CA, AB or passes through the circum-centre of $\triangle ABC$ (*vide* § 5 (3)).

Hence, if

$$\xi_0 \equiv ap\cos A + bq\cos B + cr\cos C,$$

we should be able to express ρ^2 in the form $\mu\xi_0\xi_1\xi_2\xi_3$, where μ is independent of p, q, r. This consideration and the identity (i) of § 4 enable us to obtain the result

$$\rho^2 = R^2\xi_0\xi_1\xi_2\xi_3/4M^4, \quad \ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}5)$$

where R is the circum-radius of $\triangle ABC$.

I wonder if this expression for ρ^2 is known.

8. *The envelope of straight lines whose orthopoles lie on a given straight line.*

If the orthopole of a variable line (p, q, r) lies on the given straight line $l_0x+m_0y+n_0z=0$, then from (7·3),

$$\Sigma l_0qr \tan A + (2M - \Sigma qr \tan A)(\Sigma l_0 \cot B \cot C) = 0, \quad \ldots\ldots\ldots(8{\cdot}1)$$

which is, therefore, the tangential equation of the envelope of (p, q, r). Hence the envelope is a *parabola* confocal with the parabola

$$(\Sigma l_0qr \tan A) - (\Sigma l_0 \cot B \cot C)\,\Sigma qr \tan A = 0, \quad \ldots\ldots(8{\cdot}2)$$

which touches the sides of $\triangle ABC$.

Again since the parabola (8·1) touches the line (p, q, r) and is confocal with the inconic (parabola) (8·2), the joint-orthoptic locus of the two parabolas cuts orthogonally the orthopolar circle of p, q, r, and is therefore the locus of the orthopole of the variable tangent (p, q, r) of the parabola (8·1). Hence the line (l_0, m_0, n_0) is the joint-orthoptic locus of the two confocal parabolas, and is therefore parallel to their directrices and lies *mid-way* between them.

9. *The orthopolar conic.*

The locus of orthopoles (with respect to a triangle) of straight lines through a given point P is the ellipse which passes through the vertices of the pedal triangle of P and has its centre at the middle point of the line joining P to the orthocentre H of the triangle.

Proof: Suppose $(p_1+\lambda p_2, q_1+\lambda q_2, r_1+\lambda r_2)$, where λ is a parameter, are the coordinates of a variable straight line through P. Then from (7·4), the coordinates (x, y, z) of the orthopole of this line are given by equations of the form

$$x/(x_{22}\lambda^2 + 2x_{12}\lambda + x_{11}) = y/(y_{22}\lambda^2 + 2y_{12}\lambda + y_{11}) = z/(z_{22}\lambda^2 + 2z_{12}\lambda + z_{11}),$$

which are well known to be parametric equations of a conic. Hence the locus of orthopoles is a conic.

The orthopoles of the lines through P at right angles to the sides BC, CA, AB of the triangle are obviously the vertices D, E, F respectively of the pedal triangle of P.

Again, it can be easily proved by elementary geometry that the orthopole of the line through P parallel to BC is the point D' on HA such that $HD'=DP$; similarly the orthopoles of the lines through P parallel to CA, AB are the points E', F' on HB, HC respectively such that $HE'=EP$ and $HF'=FP$. Hence the orthopolar conic passes through D, E, F, D', E', F'. Again, since DD', EE', FF' bisect each other at the middle point ω of HP, hence *ω is the centre of the orthopolar conic.* Again, since there is no real orthopole *at infinity*, the conic is an *ellipse*.

We see that the orthopolar ellipse is a circle (coinciding with the N.P. circle of $\triangle ABC$) when P is at the circum-centre O of $\triangle ABC$.

At what points does the orthopolar conic cut BC, CA, AB?

Let A_1, B_1, C_1 be the other extremities of the diameters through A, B, C respectively of the circle ABC. Let A_1P meet the circle again at A_2, and let the perpendicular from A_2 on BC meet BC at D_1; let B_2, E_1 have the same significance with respect to B_1 and CA; C_2, F_1 with respect to C_1 and AB. Then the orthopolar conic cuts BC at D, D_1; CA at E, E_1; AB at F, F_1.

When is the orthopolar conic an inconic of $\triangle ABC$?

Obviously, when $D_1 \equiv D$, $E_1 \equiv E$, $F_1 \equiv F$; that is, when P is the orthocentre of the *anti-medial* $\triangle A_1B_1C_1$ of $\triangle ABC$.

10. *Cartesian parametric equations of the orthopolar conic.*

Let us take $x^2+y^2=R^2$ as the equation of the circum-circle of $\triangle ABC$, and α, β, γ as the vectorial angles of A, B, C respectively, so that $A \equiv (R\cos\alpha, R\sin\alpha)$, and so on. Let us choose the x-axis such that $\alpha+\beta+\gamma=2n\pi$. Then if $\theta+\phi$ and $\theta-\phi$ be the vectorial angles of two points U and V on the circle ABC, we can show that the coordinates (x, y) of the orthopole W of the chord UV are given by

$$2x = x_0 + R[\cos(\theta+\phi)+\cos(\theta-\phi)+\cos 2\theta],$$

$$2y = y_0 + R[\sin(\theta+\phi)+\sin(\theta-\phi)-\sin 2\theta],$$

where $x_0 = R[\cos\alpha+\cos\beta+\cos\gamma]$,

and $y_0 = R[\sin\alpha+\sin\beta+\sin\gamma]$.

[The use of complex numbers will simplify the work.]

If UV passes through a fixed point $P \equiv (d\cos\psi_0, d\sin\psi_0)$, then

$$d\cos(\theta-\psi_0) = R\cos\phi.$$

Hence the coordinates (x, y) of the orthopole of a *variable* line through a fixed point P can be expressed in terms of the parameter θ in the form

$$2x = x_0 + 2d\cos\theta\cos(\theta-\psi_0) + R\cos 2\theta,$$

$$= x_0 + d\cos\psi_0 + d\cos(2\theta-\psi_0) + R\cos 2\theta,$$

and $$2y = y_0 + d\sin\psi_0 + d\sin(2\theta-\psi_0) - R\sin 2\theta.$$

If we now transform the origin to the point

$$\omega \equiv [\tfrac{1}{2}(x_0+d\cos\psi_0), \tfrac{1}{2}(y_0+d\sin\psi_0)],$$

the coordinates (X, Y) of the orthopole W are given by

$$X = \tfrac{1}{2}d\cos(2\theta-\psi_0) + \tfrac{1}{2}R\cos 2\theta,$$

$$Y = \tfrac{1}{2}d\sin(2\theta-\psi_0) - \tfrac{1}{2}R\sin 2\theta,$$

where θ is a parameter. .....(10·1)

We see readily that these are parametric equations of an ellipse *whose centre is at the new origin* ω, which is at the middle point of HP, since $H \equiv (x_0, y_0)$ with reference to the original axes of co-ordinates.

If $\omega W = r$, then

$$r^2 = X^2 + Y^2$$
$$= \tfrac{1}{4}[R^2 + d^2 + 2dR\cos(4\theta - \psi_0)],$$

which is *maximum* when $\cos(4\theta - \psi_0) = 1$, and *minimum* when $\cos(4\theta - \psi_0) = -1$. Hence *the lengths of the major and minor axes are*

$$R + d \quad \text{and} \quad R - d, \qquad \text{.....(10·2)}$$

and their equations are

$$X \sin \tfrac{1}{2}\psi_0 + Y \cos \tfrac{1}{2}\psi_0 = 0,$$

and

$$X \cos \tfrac{1}{2}\psi_0 - Y \sin \tfrac{1}{2}\psi_0 = 0$$

respectively. .....(10·3)

Though I have not seen anywhere the results of this section, their very simplicity makes me suspect that they cannot be unknown.

From the above we see also that *the auxiliary circle of the orthopolar conic touches the* N.P. *circle of* $\triangle ABC$.

11. *A new theorem—an extension of Feuerbach's Theorem :*

If P be a point on a conic Y confocal with an inconic X of a triangle ABC, then the pedal circle of P with respect to the $\triangle ABC$, *the joint-director circle of X and Y, and the director circle of the inconic of* $\triangle ABC$ *which touches Y at P are coaxal ; and when P and ABC are fixed while X varies, the envelope of the radical axis of the aforesaid circles is the orthopolar conic of P with respect to* $\triangle ABC$.

Proof : If PU, PV are tangents from P to a conic Y confocal with an inconic X of $\triangle ABC$, and X' be the inconic which touches PU and PV, then the orthopolar circles of PU and PV cut at right angles

(i) the pedal circle of P,

(ii) the joint-director circle of X and Y,

and (iii) the director circle of X'.

Hence the circles (i), (ii) and (iii) are coaxal, the common radical axis being the chord of the orthopolar conic of P joining the orthopoles of PU and PV.

If we take the particular case when P lies on Y, that is, when PU and PV coincide, we get the theorem enunciated above.

It will be seen that the theorem stated by Prof. L. J. Rogers in *Note* 953 (*Math. Gazette*, No. 207) is a particular case of my general theorem of this section. It should also be pointed out that the particular case was first published by the late V. Ramaswami Aiyar of Madras in the Edinburgh Mathematical Society's *Proceedings*, 1896.

12. *Another extension of Aiyar's Theorem.*

If σ *be the joint-director circle of an inconic X of a triangle ABC and a confocal Y, then the radical axis of* σ *and the* N.P. *circle of the triangle cuts the latter at an angle equal to that subtended by Y at the circum-centre of the triangle.*

This theorem can be proved with the aid of the following principles :

(i) The radical axis of σ and the N.P. circle cuts the latter at the orthopoles of the tangents OT, OT' from O to Y.

(ii) *The angle between any two diameters of the circle ABC is equal to half the angle which their orthopoles subtend at the centre of the* N.P. *circle* [use (10·1) when $d=0$].

The particular case of the above general theorem when $Y \equiv X$ was discovered and published by me in 1921 in the *Journal of the Indian Mathematical Society*, Vol. XIII, p. 217. The extension was suggested by Mr. M. Bhimasena Rao of Mysore. Aiyar's theorem, reintroduced by Prof. Rogers, is obviously a particular case of the above when $OT' \equiv OT$.

13. *Coaxal pedal circles :*

(1) Let (P_1, P_1'), (P_2, P_2') be two pairs of isogonal conjugate points with respect to a triangle ABC. Let P_1P_2', $P_1'P_2$ meet at P_3, and P_1P_2, $P_1'P_2'$ meet at P_3'. Then the *areal* envelope equations of the two point-pairs (P_1, P_1') and (P_2, P_2') with respect to $\triangle ABC$ are of the form

$$\psi \equiv (x_1l + y_1m + z_1n)(a^2l/x_1 + b^2m/y_1 + c^2n/z_1) = 0,$$

and

$$\psi' \equiv (x_2l + y_2m + z_2n)(a^2l/x_2 + b^2m/y_2 + c^2n/z_2) = 0.$$

Hence the equation of any inconic of the quadrilateral $P_1P_2P_1'P_2'$ is

$$\lambda\psi + \mu\psi' = 0 \quad \text{.....(13·1)}$$

where λ/μ is a parameter.

Now corresponding to every conic of the system $\lambda\psi + \mu\psi' = 0$, there exists a confocal inconic

$$\lambda\psi + \mu\psi' - (\lambda + \mu)\,\Omega = 0 \quad \text{.....(13·2)}$$

of $\triangle ABC$.

The point-pair (P_3, P_3') is a degenerate conic-envelope of the system $\lambda\psi + \mu\psi' = 0$, and hence there exists an inconic of $\triangle ABC$ which has (P_3, P_3') for foci.

Again, corresponding to the value $\lambda/\mu = -1$, we have the unique conic S of the system $\lambda\psi + \mu\psi' = 0$ which is an inconic of the quadrilateral $P_1P_2P_1'P_2'$ and of $\triangle ABC$. We have thus the following result :

If two pairs of opposite angular points of a complete quadrilateral are isogonal pairs with respect to a triangle, so are the third pair ; and a conic can be drawn to touch the sides of the quadrilateral and those of the triangle. .....(13·3)

(2) Again for all values of λ/μ, the joint-director circles of the confocals given by (13·1) and (13·2) cut orthogonally the orthopolar circles of P_1P_2, P_2P_1', $P_1'P_2'$, $P_2'P_1$. Hence the joint-director circles are members of a coaxal system (Γ), while the orthopolar circles are members of the orthogonal coaxal system (Γ'). To the former system (Γ) belong the pedal circles of (P_1, P_1'), (P_2, P_2'), (P_3, P_3')

with respect to ΔABC, and the director circle of conic S of (1) above; the radical axis of the system contains the orthopoles of P_1P_3, P_2P_1', $P_1'P_2'$, $P_2'P_1$, and is the joint-orthoptic locus of the parabola which touches the sides of the quadrilateral $P_1P_2P_1'P_2'$ and the confocal inconic of ΔABC.

Further, it can be shown that there cannot be a fourth isogonal pair (P_4, P_4') whose pedal circle with respect to ΔABC is coaxal with those of P_1, P_2, P_3. For, otherwise, the lines P_1P_2, P_1P_3, P_1P_4 will all touch the same parabola (§ 8), which is impossible.

Hence the necessary and sufficient condition for the pedal circles of P_1, P_2, P_3 to be coaxal is that the isogonal point-pairs (P_1, P_1'), (P_2, P_2'), (P_3, P_3') should be the three pairs of opposite angular points of a complete quadrilateral. .....(13·4)

(3) Again for the value of λ/μ which makes the coefficient of mn vanish in the L.H.S. of (13·2), we have a conic of the system $\lambda\psi + \mu\psi' = 0$ such that the confocal inconic of ΔABC is a point pair, viz. its foci which are at A and at a point A_0 on BC. Similarly we have a conic of the system having its foci at B and at a point B_0 on CA; and a third conic with its foci at C and at a point C_0 on AB.

Hence we have the following results:

(i) (A, A_0), (B, B_0), (C, C_0) are isogonal pairs with respect to each of the triangles $P_1P_2P_3$, $P_1P_2'P_3'$, $P_1'P_2P_3'$, $P_1'P_2'P_3$.

(ii) (A, A_0) have the same pedal circle with respect to each of the triangles of (i) above; and the feet of the perpendiculars from A on the sides of the quadrilateral $P_1P_2P_1'P_2'$ lie on that pedal circle. Similarly for the pedal circles of (B, B_0), (C, C_0).

(iii) The pedal circles of (A, A_0), (B, B_0), (C, C_0) with respect to the four triangles are joint-director circles of the coaxal system (Γ) of (2) above.

(iv) $A_0B_0C_0$ is a straight line. (This follows from (iii) and (13·4) above.)

(v) *The quadrilateral formed by the straight lines BC, CA, AB and $A_0B_0C_0$ is related to $\Delta P_1P_2P_3$ exactly in the same manner as the quadrilateral $P_1P_2P_1'P_2'$ is to ΔABC; and hence the conic S of* (13·3) *above touches the sides of the two quadrilaterals.*

(vi) Since the triangles ABC and $P_1P_2P_3$ are circumscribed to the conic S, they must be inscribed in a conic S', where S and S' are so related that S touches the isogonal transformation of S' with respect to either triangle.

From the foregoing we deduce the following interesting theorem, which, I believe, is new:

If the pedal circles of the vertices of a triangle Δ_2 with respect to a triangle Δ_1 are members of a coaxal system, then the pedal circles of the vertices of Δ_1 with respect to Δ_2 are also members of the same coaxal system. The necessary and sufficient condition that the two triangles should be so related is that they should be inscribed in a conic S' and circumscribed to a conic S which touches the isogonal transformation of S' with respect to either triangle.

14. *Concurrent pedal lines.*

In the foregoing, when the conic S' is the circle ABC, its isogonal transformation with respect to $\triangle ABC$, viz. the line $P_1'P_2'P_3'$ becomes the line at infinity; likewise the line $A_0B_0C_0$; hence the conic S becomes a parabola. We are thus able to deduce, as a particular case of the preceding theorem, the now well-known property of concurrent pedal lines. This may be deduced independently from the equations of (7·1) as in the following section.

15. *The three straight lines having a given orthopole W (x, y, z) with respect to a triangle ABC.*

If W be the orthopole of the line (p, q, r), then taking $x+y+z=1$, we have from (7·1),

$$D_{BC} - qr = D_{CA} - rp = D_{AB} - pq.$$

Hence

$$p(q-r) + 2M(y \cot B - z \cot C) = 0,$$

or
$$2Mp(q-r) + (y \cot B - z \cot C)\,\Omega(p, q, r) = 0. \quad \ldots\ldots(15\cdot1)$$

Similarly,

$$2Mq(r-p) + (z \cot C - x \cot A)\,\Omega(p, q, r) = 0, \quad \ldots\ldots(15\cdot2)$$

and
$$2Mr(p-q) + (x \cot A - y \cot B)\,\Omega(p, q, r) = 0. \quad \ldots\ldots(15\cdot3)$$

From (15·1), (15·2) and (15·3), we get

$$\Sigma(y \cot B - z \cot C)\,qr = 0. \quad \ldots\ldots(15\cdot4)$$

Hence given (x, y, z), the line (p, q, r) is a common tangent of the four parabolas represented by (15·1), (15·2), (15·3), (15·4).

There are three such common tangents, and hence there exist three straight lines whose orthopoles are at W. [It will be sufficient to take the parabolas (15·1) and (15·2) and draw their common tangents.]

The foci of the parabolas of (15·1), (15·2) and (15·3) are (A, A'_0), (B, B_0'), (C, C_0'), where A_0', B_0', C_0' are the points at infinity on BC, CA, AB respectively. Hence, if the three common tangents form a triangle $P_1P_2P_3$, then the foci A, B, C of the three parabolas lie on the circle $P_1P_2P_3$. Hence the triangles ABC, $P_1P_2P_3$ are inscribed in the same circle, and from (15·4) they are circumscribed to the same parabola.

Again, if H, H' be the orthocentres of the triangles ABC, $P_1P_2P_3$, by considering the joint-orthoptic locus of the parabola (15·1) and the point-pair $p(q-r)=0$, we can show that W lies on the line through the middle point of HH' parallel to AH (and hence at right angles to BC). Similarly W lies on the line through the middle point of HH' at right angles to CA. *Hence W is at the middle point of HH'.*

The pedal lines of P_1, P_2, P_3 with respect to $\triangle ABC$, and the pedal lines of A, B, C with respect to $\triangle P_1P_2P_3$ meet at W.

F. H. V. GULASEKHARAM

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CORRESPONDENCE.

The Hamiltonian Revival.

To the Editor of the *Mathematical Gazette.*

Sir,—Professor E. T. Whittaker's courtesy in answering the questions I raised recently relating to a sentence in his article "The Hamiltonian Revival" is deeply appreciated. At the same time many who use the calculus of vectors and dyadics will regret to find in Professor Whittaker's letter little realisation of its beauties and powers. Professor Whittaker approaches the question as a pure mathematician interested in "any set of objects of thought which may be called *generalised numbers* provided they satisfy certain definitions" concerning equality, addition and multiplication and form a "group". But he shows no recognition of the point of view of the natural philosopher who wishes to deal with physical phenomena in the universe in which we live. This ignoring of the needs which presumably inspired Gibbs and Heaviside pervades much of what Professor Whittaker has to say, in both his article and his letter, on the subject of vectors.

In his article he condemned dyadics and vector-analysis as reproducing "the limitations imposed by (their) close association with the geometry of ordinary space". But it is precisely because vector analysis embodies the quality of "three-dimensionality" that it is so appropriate to the handling of physical problems. In *tensor* analysis the "vector product" of two vectors A_μ and B_ν requires for its expression the use of the "alternate tensor", of rank n equal to the number of dimensions of the space being used; the resulting tensor is of rank $(n-2)$. In three-dimensional space $(n=3)$, the resulting tensor is itself of rank 1, *i.e.* is a vector; hence in *vector* analysis the "vector product" of two vectors is so-called because amongst the various products of two vectors which can be constructed this one is also a vector. Hence arises the power of vector analysis proper, a power which tensor analysis in general, whether for "flat" space or for a general Riemannian space, exercises only by an excessive use of suffixes. Professor Whittaker objects to the calling of $\mathbf{B}$ a true product in the equation $\mathbf{A}\wedge\mathbf{X}=\mathbf{B}$, because the vector $\mathbf{X}$ is not uniquely determined by $\mathbf{A}$ and $\mathbf{B}$. But multiplying each side vectorially by $\mathbf{A}$ and using the "continued vector product" theorem (which may be regarded as the "signature-tune" of ordinary space), we have

$$-\mathbf{A}(\mathbf{X}\,.\,\mathbf{A})+\mathbf{X}\mathbf{A}^2=\mathbf{B}\wedge\mathbf{A}$$

or

$$\mathbf{X}=\frac{\mathbf{B}\wedge\mathbf{A}}{\mathbf{A}^2}+\lambda\mathbf{A},$$

where λ is an arbitrary constant. Does Professor Whittaker condemn differential equations as not "true" equations because their solution involves arbitrary constants? The component $\lambda\mathbf{A}$ of $\mathbf{X}$ is arbitrary in modulus because this component is irrelevant in many physical applications. Mechanics, in vector presentation, turns largely on the use of the two relations $\mathbf{L}=\mathbf{r}\wedge\mathbf{P}$ and $\mathbf{v}=\boldsymbol{\Omega}\wedge\mathbf{r}$, where $\mathbf{L}$ is the moment about the origin of a force $\mathbf{P}$ acting at $\mathbf{r}$, and $\mathbf{v}$ is the velocity at $\mathbf{r}$ associated with an angular velocity $\boldsymbol{\Omega}$ of the rigid body of which the point located at the origin is at rest; in these relations, the component of $\mathbf{P}$ or $\boldsymbol{\Omega}$ along $\mathbf{r}$ is physically irrelevant, and the vector-product is useful precisely because it seizes on the relevant and rejects the irrelevant.

To require a combination of two entities to be what Professor Whittaker calls a "true product" ignores that two given entities may have many different multiplicative combinations, whether we call them "products" or

not; *e.g.* two vectors **A**, **B** have the combinations **A . B**, **A∧B** and the dyad-product **AB**, which form the hierarchy scalar, vector and tensor of rank two. The flexibility of the calculus invented by Gibbs arises from a wise employment of these different "products", and in the ordinary space most used by physicists these products may readily be handled without the use of suffixes.

In this connection I may refer to Professor Whittaker's statement that Gibbs and Heaviside "did not rise to the idea of treating vectors by means of a calculus of generalised numbers of a new type: all they wanted was some way of writing ordinary Cartesian analysis without putting the axes of co-ordinates in evidence". In the first place, their calculus *was* new, on Professor Whittaker's own showing, in that it departed from the group property; in the second place, it is necessary to distinguish between the frame of reference, the axes of coordinates and the components of vectors with respect to these axes. The frame of reference—the observer's observational apparatus—must always be in evidence, though the pure mathematician may cloak it in a mantle of invisibility; particular axes of reference have certain conveniences when specific directions are of interest; the components of vectors with respect to arbitrary axes have little physical interest. *Tensor* calculus is, indeed, a condensed method of writing out relations between numerical components or coordinates; it always mentions coordinates, but avoids mentioning each one separately by the magic of its summation convention. In *vector* analysis we suppress suffixes altogether: $P_\mu Q_\mu$ becomes **P . Q**, $P_\mu Q_\nu$ becomes **PQ**, $E_{\alpha\mu\nu}P_\mu Q_\nu$ becomes the α component of **P∧Q**; and we deal directly with the entities represented by the symbols, not their components. It is *not* a "syncopated form of ordinary Cartesian analysis". The power of vector analysis, again, is seen in the circumstance that we can pass readily to the suffix notation if for the moment it proves more convenient. But vector analysis has the further advantage that it can pick out axes of physical interest which are not mutually perpendicular, and yet retain the advantages of an orthogonal set of axes.

For example, in handling the motion of a top, vector analysis concentrates attention on two directions of interest, the axis of the top and the vertical. It needs to mention no others. The immense logical superiority of vector analysis over Cartesian is evident when we reflect on that perversion by which in some text-books, the laws of nature are formulated in vector form, embodied in Cartesian equations, applied to a sleeping top, recombined to give equations in a complex variable $x+iy$, the resulting differential equation in z solved, and z re-dissected into its component scalars, which then need to be mentally recombined to give the physical kinematic picture. In the solution of a problem by the methods of vector analysis, there is no need for an eventual interpretation of the final result: the problem is formulated vectorially and solved vectorially, and the vectorial solution proclaims the resulting motion.

In conclusion, if quaternions are so useful and natural a mathematical tool, why are they not habitually employed in research and teaching?

Yours truly, E. A. MILNE.

To the Editor of the *Mathematical Gazette*.

SIR,—What I tried to show in my previous letter was that vector-analysis has certain inherent defects which must always prevent it from developing into a satisfactory calculus. It is not difficult to illustrate this, for example, on the ground of Professor Milne's own choosing, namely the different multiplicative combinations. If the vector-analyst forms the different kinds of products of three vectors, such as **(A∧B) . C**, he has a choice of 3 alternatives

for the first multiplicative sign, 3 for the second multiplicative sign, 2 for the position of the brackets denoting association, and 6 for the permutations of A, B, C, that is, he has $3 . 3 . 2 . 6$ or 108 formally different products of 3 vectors, of which, however, many are meaningless, many equal to each other, and many equal to each other with sign reversed. Yet with all this multitude of forms, there is no vector-analytic expression, formally symmetric in the 3 vectors, which represents the simplest symmetric function of the vectors, namely the volume of the parallelepiped of which they are three edges. How much simpler is the quaternion theory, where there is only one product of the three vectors in a given order. It is a quaternion, and the volume of the parallelepiped is its scalar.

Between the many forms there are many identical relations. But how much better to do away with both!

Professor Milne suggests that quaternions are less physical than vector-analysis. But any vector-analysis solution of a problem can be at once transliterated into quaternions, the two solutions differing merely in notation. The converse however is not true, since the powerful concept of division by a vector, which is used in quaternions, does not exist in vector-analysis.

At the end of his letter, Professor Milne puts the interesting question, why quaternions have not been used habitually by physicists instead of vector-analysis. The reason is, I think, to be found in the combination of two circumstances. Firstly, that the problems which physicists solve by vector-analysis are practically always of a linear or quadratic character, and therefore so simple from the mathematical point of view that the defects of vector-analysis don't cause any trouble : and secondly, that vector-analysis can be picked up in an hour or so, involving as it does no methods radically different from those of Cartesian analysis, whereas the quaternion calculus is to the physicist a new kind of pure mathematics, which he is reluctant to learn. I am inclined to conjecture that if the present range of problems were not extended, physicists would continue to prefer vector-analysis ; but that the development of relativity and quantum mechanics will sooner or later require quaternion methods for the more difficult problems (as in Professor Conway's paper, referred to in my original article) ; and that when the physicists have thus been forced to learn quaternions, they will use them for all purposes, and vector-analysis will be forgotten. Yours truly, E. T. WHITTAKER.

[This correspondence is now closed.—Ed.]

THE MATHEMATICIAN AND THE COMMUNITY.

To the Editor of the *Mathematical Gazette*.

SIR,—In *A Mathematician's Apology*, Professor Hardy writes : "There is no instance, so far as I know, of a first-rate mathematician abandoning mathematics and attaining first-rate distinction in any other field. There may have been young men who would have been first-rate mathematicians if they had stuck to mathematics, but I have never heard of a really plausible example. And all this is fully borne out by my own very limited experience. Every young mathematician of real talent whom I have known has been faithful to mathematics, and not from lack of ambition but from abundance of it ; they have all recognized that there, if anywhere, lay the road to a life of any distinction."

It is easy to maintain this argument by simply denying that any other form of success is comparable with a mathematician's in his own line. That mathe-

maticians themselves have been generally convinced of it seems very far from true. It is only necessary to think of the seventeenth century, when mathematicians had the time of their lives; though it took them another century before they realized it.

We have only to think of Des Cartes, Pascal, Barrow, Wren, Newton and Leibniz to see that there have been mathematicians, and eminently serious ones too, who have not thought it necessary, and have even thought it wrong, to devote the whole of their lives to the pursuit of mathematics. In fact, in the seventeenth century, John Wallis was an exception; he seems to approach nearest to the modern type of a professional mathematician.

The question naturally arises whether it is better to seek fame as a mathematician or to serve one's day and generation in some other way. Whether it is better to rectify the cycloid or to build St. Paul's Cathedral? The choice comes to few, but Professor Hardy's answer is not altogether convincing. If to-day the choice seems to be made easily the explanation lies near at hand in two facts. In the first place, given the requisite ability of a special kind, the path of the mathematician through scholarships and fellowships is ridiculous in its simplicity and it leads automatically to its end. In the second place, before we are impressed by the small number of desertions it may be well to ask, what are the opportunities and the temptations to desert? At the present day they seem to be negligible, and it is possible that great talents capable of serving the interests of society in manifold ways are confined within too narrow a track. Probably Professor Hardy is right about the greatest mathematicians; they serve their age best by being great mathematicians. But there are others not quite on that exalted level, yet possessing gifts by no means contemptible. That all of them should be caught up and retained in the academic machine may be neither fair to them nor good for the community at large.

The state of things which Professor Hardy complacently regards as normal is really the artificial product of a peculiar social system. The real mathematician needs no apology. But there is another side to the question. Mathematical talent (setting aside genius) is a gift by no means too common. Is society wise in its own interest to encourage a system which absorbs all those who possess this valuable gift of general utility and treats them as so much academic fodder to be made into specialized mathematicians? This is the present tendency and the community definitely suffers by it.

Yours, etc., H. C. PLUMMER.

1381. "But it's my job to weigh evidence," said Harriet, "and I can't help seeing the strength of the police case. It's a matter of $a+b$, you know. Only there happens to be an unknown factor."

"Like that thing that keeps cropping up in the new kind of physics," said the Dean. "Planck's constant, or whatever they call it."

An hour later they were still at it. Finally the Bursar was heard to quote: "God made the integers; all else is the work of man."

"Oh, bother!" cried the Dean. "Do let's keep mathematics out of it. And physics. I cannot cope with them."

"Who mentioned Planck's constant a little time ago?"

"I did, and I'm sorry for it. I call it a revolting little object."—D. L. Sayers, *Gaudy Night*, pp. 37 and 40. [Per Mr. F. W. Kellaway.]

MATHEMATICAL NOTES.

1555. *Approximations to roots.*

If a is an approximation to the cube root of N, a *much* closer approximation is given by

$$a\left\{\frac{10N+a^3+16a\sqrt{(aN)}}{N+10a^3+16a\sqrt{(aN)}}\right\}.$$

If a is an approximation to the fifth root of N, a *much* closer approximation is given by

$$a\left\{\frac{7N+2a^5+16a^2\sqrt{(aN)}}{2N+7a^5+16a^2\sqrt{(aN)}}\right\}.$$

Although the square root embodied in the formulae is unsatisfactory, its appearance is compensated by the remarkable results. Thus if $N=10$, $a=2$, the approximation to $\sqrt[3]{N}$ is correct to 7 decimal places; if $N=10$, $a=2{\cdot}15$, the approximation is correct to 16 decimal places.

Similar remarkable accuracy is obtained for the fifth root.

A rapidly calculated formula for the square root is

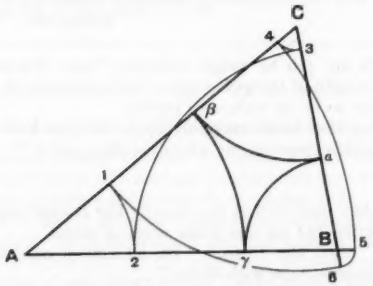
$$\sqrt{N}=a+\frac{(N-a^2)(N+3a^2)}{4a(N+a^2)},$$

the error being $h^4/4a(N+a^2)$ where $N=(a+h)^2$. As an illustration, take $N=10$, $a=3$. The approximation gives $\sqrt{N}=3{\cdot}16228$, correct to five decimal places.

R. H. BIRCH.

1556. *Equidiametral curves.*

The discussion of equidiametral curves (Gleaning 1325, *Gazette*, October 1940, and Note 1516, *Gazette*, May 1941) suggests the construction shown in the diagram for the solution of the equations



$x+y=a$, $y+z=b$, $z+x=c$, where a, b, c are the sides of a triangle, that is, to draw three circles whose centres are the vertices of a triangle and which are such that each touches the other two.

Draw the arcs 12 with centre A, 23 with centre B, and so on; these

arcs are all drawn clockwise if the points A, B, C are taken anti-clockwise. Then if α, β, γ are the midpoints of the segments 36, 14, 25 respectively, the arcs $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ satisfy the required conditions.

My pupil, Mr. A. B. Partridge, has pointed out that this construction can be extended to any polygon with an odd number of sides, but that for a polygon with an even number of sides the problem is either insoluble or indeterminate. Thus, for example, a quadrilateral must be circumcyclic to provide a solution, and then the number of solutions is infinite. B. A. SWINDEN.

1557. *Another all-integral cyclic hexagon.* (See Note 1514.)

Sir Richard Harington has very kindly drawn my attention to a Pythagorean triplet:

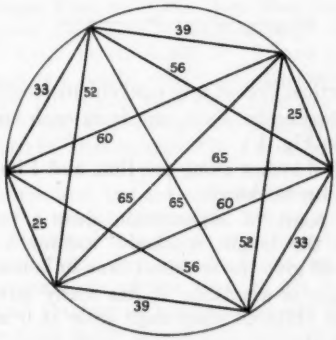
$$\begin{aligned} 65^2 &= 56^2 + 33^2 \\ &= 60^2 + 25^2 \{ = 5^2(12^2 + 5^2)\} \\ &= 52^2 + 39^2 \{ = 13^2(4^2 + 3^2)\}. \end{aligned}$$

Trying the first two relationships and dealing with them as in my Note 1514 (*Gazette*, May 1941) yielded no immediate results, but did give these relationships with the third line:

$$(33 \times 25) + (39 \times 65) = 60 \times 56,$$

$$(56 \times 25) + (33 \times 60) = 52 \times 65.$$

By a slightly different route, these five equations taken together give a cyclic hexagon possessing quite a different form of symmetry from that of the hexagon in Note 1514.



By way of a check, we find also that

$$(33 \times 39) + (25 \times 65) = 56 \times 52,$$

and

$$(25 \times 39) + (33 \times 65) = 52 \times 60.$$

This gives an array of strange coincidences which suggest that

both this and the figure in Note 1514 are but the first members of a series that could be expressed in terms of one or two relatively simple formulae. Can anyone find these?

C. DUDLEY LANGFORD.

1558. *A graphical method of solving problems on "Rate of Work" and similar problems.*

I am afraid that I cannot be certain of the originality of this method. I thought that I had found it in a textbook used by my father or grandfather but as I have never been able to find it again, and as few people seem to know it, perhaps my impression that I had seen it in a book is wrong.

The method of teaching it must of necessity be varied according to both the teacher and the class; but the following account gives, briefly, the method I have found best for children of about 11 and 12. Actually children of 9 or even of 8 love the simple first case, but the approach must then be somewhat modified.

I start off by showing the class how delightfully simple the method is and then show why it works. Their interest is fully aroused by the fact that such a problem can be done so easily and they are all eager to hear about it.

If one man takes 4 hours to do a piece of work and another takes 6 hours, how long will they take if they work together?

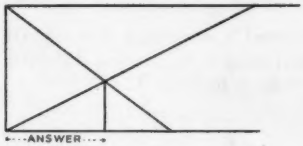


FIG. 1.

(i) Draw a vertical line of any convenient length, say, 3 inches;

(ii) draw two horizontal lines, one from each end of the vertical line, as in Fig. 1;

(iii) measure off 6 inches along one line, and 4 inches on the other and join up as shown;

(iv) from the point of intersection drop a perpendicular and measure the length marked "answer". The length in inches will give the required time in hours.

If squared paper is available, it has many advantages, but we must see that the children start their scale at 0 and not at 1, as they are so apt to do.

The explanation is simple, though difficult to put briefly on paper. Suppose the men are paving a path. Draw a line, in green chalk, to represent the path at the beginning. Then one inch to the right draw another to represent the path after one hour; then one more inch to the right another to represent the path after two hours, and

so on. Now think of the man who takes 6 hours. He would start at one end and get to the other at the end of 6 hours and so we change the last upright line from green to brown. In three hours he would have got half-way and so half of the appropriate upright is changed to brown ; with a little discussion the class can see that the amount to be changed from green to brown on any line can be found by drawing an oblique straight line, as in Fig. 2.

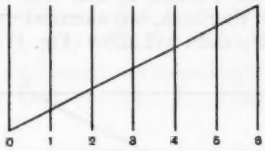


FIG. 2.

Now suppose a second man, who alone would take 4 hours on the job, comes to help. So as not to get in the way, he starts at the other end of the path and similar reasoning brings the class to Fig. 3.

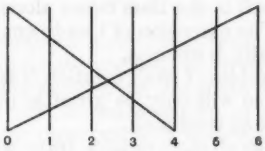


FIG. 3.

When the oblique lines meet the men have met and the job is finished. Not much discussion is required to get from Fig. 3 to the more convenient form shown in Fig. 1 ; I have found it helpful to spend longer than one might expect on discussing lines 1 and 2, showing how they tell which part each man has done, and which part is still to be done, and seeing that none thinks that the path runs from left to right, that is, that Fig. 1 is in some way a drawing of the path.

Pupils see at once that the same method would do for any kind of work, though the explanation might not be so easy. (I believe that it has been suggested that if two taps were filling a tank, then it should be supposed that the water from one tap stayed at the top of the tank. Explanations of this kind, to my mind, only make a pupil think that all teachers of mathematics are mad.)

Pupils also see quite easily that if four men are at work, they can be best taken in pairs, and then the times for these pairs combined as if they represented two men.

Other problems can be dealt with by this method. Thus, in optics, many problems can be reduced to the equation

$$(1/a) + (1/b) = 1/x,$$

where a, b, x are all positive. This is exactly the situation which Fig. 1 will solve, for if the times taken by the men are m and n, and the total time is t, then $t=mn/(m+n)$, whereas in the equation above $x=ab/(a+b)$. The rest is obvious.

Another problem which can be solved by the method of Fig. 1 is of this type. Given the time taken by one man alone and also the time when he is helped by a second man, find the time which the second man would take working alone. The class discussion must be rather lengthy and thorough, but summed up it is simple. Draw as much of Fig. 1 as the data will allow (Fig. 4) and then see how we

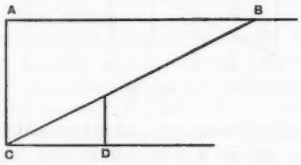


FIG. 4.

could complete it. AB is the time taken alone, CD the time when both are working. The extension of this to enable optical problems to be solved graphically is obvious.

The last type of problem I shall mention is far less easily grasped by most pupils and so will only be given in outline to those most interested, to expand as they wish.

If a kitchen help fills all the available trays with bread and butter in 15 minutes, if none is used; and if the waitress can put them all on the tables and return them empty in 20 minutes if none is refilled; then how long will it be before all are filled if both are at work together. (Teachers will probably recognise an old friend, the bath with the tap and leaking plug.)

Suppose the trays, when full, are put into a rack, one below the other, starting at the top. They will be taken out at the other side, also starting at the top so as always to take first that whose contents have been buttered longest. We thus get Fig. 5.

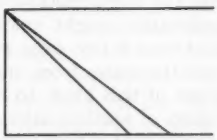


FIG. 5.

Two alternatives are then possible. Suppose, as would be the case in practice, that when either got to the bottom of the rack, she started at the top. We then get Fig. 6. We here actually find when and how the kitchen help catches up the waitress. (The lines

marked *KH* are those of the kitchen help, those marked *W* for the waitress.)

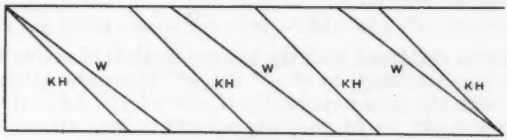


FIG. 6.

The second method, more mathematical in theory, but less accurate in practice, is to get from Fig. 5 to Fig. 7 by a careful discussion.

FIG. 7.

These graphical methods have been found of use by engineers requiring quick approximate answers, by lively youngsters of 8 and 9 and by older pupils, particularly by those who seemed to think that a thing could not be mathematics at all if it did not fit nicely into one of their three water-tight compartments, arithmetic, algebra or geometry.

Obviously the first type of problem is of more utility than the other two, not only for ease of comprehension, but also in accuracy of the result. This graphical method is obviously very suitable for the problem of the man who comes to help another who has already been at work for an hour or more. C. DUDLEY LANGFORD.

1559. *On Note* 1514.

Here are some more examples in integers to satisfy the request in Note 1514, where it was stated that

(1) $25^2 = 20^2 + 15^2 = 24^2 + 7^2$, where also $25 \times 20 = 20 \times 7 + 15 \times 24$
(or $24 \times 20 = 15 \times 7 + 15 \times 25$).

It is also true that

(2) $169^2 = 13^4 = 156^2 + 65^2 = 120^2 + 119^2$ with
$169 \times 156 = 156 \times 119 + 120 \times 65.$

(3) $289^2 = 17^4 = 255^2 + 136^2 = 161^2 + 240^2$ with
$289 \times 255 = 255 \times 161 + 240 \times 136.$

Further examples would be 29^4, 37^4, 41^4, *i.e.* biquadrates of prime numbers of the form $4n + 1$, if we omit obvious cases like 50^2, 100^2, etc., derived from the examples already given.

The problem to divide a square into the sum of two squares in two different ways was first solved by Diophantos who gave a numerical example of the identity

$$(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ac-bd)^2+(ad+bc)^2.$$

This formula, combined with the ancient method of giving the sides of a right-angled triangle by a^2+b^2, $2ab$, a^2-b^2 permits us to give four triangles with the same hypotenuse $H=(a^2+b^2)(c^2+d^2)$; if we allow for the case $a=kc$, $b=kd$, they are reduced to two, viz.

$$H,\quad A=2ab(c^2+d^2),\qquad B=(a^2-b^2)(c^2+d^2);$$

$$H,\quad C=2(ac-bd)(ad+bc),\quad D=4abcd-(a^2-b^2)(c^2-d^2),$$

with $$H^2=A^2+B^2=C^2+D^2.$$

If we demand further that $HA=BC+AD$, then we get

$$ab(c^2+d^2)=cd(a^2+b^2),$$

which is true for

$$a=c,\ b=d\ \text{ (or } a=kc,\ b=kd).$$

We then have $H=(a^2+b^2)^2$,

$$A=2ab(a^2+b^2),\qquad B=(a^2-b^2)(a^2+b^2),$$
$$C=4ab(a^2-b^2),\qquad D=4a^2b^2-(a^2-b^2)^2.$$

In the examples we have :

(1) $H=25=(2^2+1^2)^2$, $\quad a=2,\ b=1.$

(2) $H=169=(3^2+2^2)^2$, $\quad a=3,\ b=2.$

(3) $H=289=(4^2+1^2)^2$, $\quad a=4,\ b=1.$

One sees easily that further examples will be given by $a=5$, $b=2$; $a=6$, $b=1$; $a=5$, $b=4$, etc., when $H=p^2$, with $p=4n+1$.

In his notes to Diophantos (III, 22) Fermat states that (*a*) a prime of the form $4n+1$ and its square is the sum of two squares in one way; its cube and biquadrate in two ways, etc.: (*b*) if a prime which is the sum of two squares be multiplied into another prime which is also the sum of two squares, the product will be the sum of two squares in two ways, etc. He also shows (*c*) how to find a number which shall be the sum of two squares in an assigned number of ways.

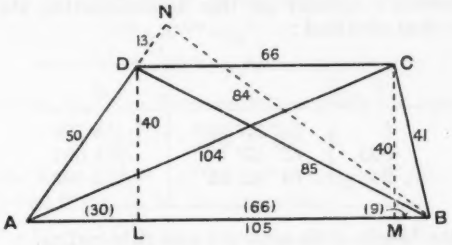
The fundamental identity was first written down in general terms by Vieta (before 1591) who also saw the connection with Ptolemy's theorem, *i.e.* with the addition theorem of the trigonometric functions. Vieta followed Diophantos in admitting rational solutions to problems of number theory where Fermat later on insisted on integers " to please his friend Frénicle de Bessy who is so very fond of integers ".

A. PRAG.

1560. *A whole-number trapezium.*

To construct a trapezium $ABCD$ having no two sides equal or perpendicular to each other, such that the four sides, the diagonals AC, BD, and the perpendicular distances $D(AB)$, $B(DA)$ are all integral.

Draw any line AB of length 105, and on it make $AL=30$, $LM=66$; then $MB=9$.



From L, M set up LD, MC perpendicular to AB, each of length 40. Then $BC=\sqrt{1681}=41$; $DA=\sqrt{2500}=50$; $CD=66$.

Also
$$AC=\sqrt{(96^2+40^2)}=104;$$
$$BD=\sqrt{(75^2+40^2)}=85.$$

Produce AD to N so that $AN=63$. Then since
$$105^2-63^2=85^2-13^2=84^2,$$
BN is perpendicular to DA and is of length 84.

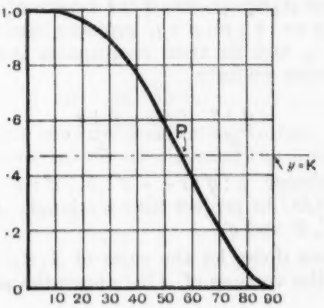
The given integers appear to be the least possible.

G. WOTHERSPOON.

1561. *On Note* 1477.

After reading Mr. Pedoe's note on the Donkey-Field problem (*Gazette*, Oct. 1940, Note 1477) I decided to set the problem to some pupils and was thus constrained to obtain solutions of the equation
$$2\alpha\cos 2\alpha-\sin 2\alpha+\pi(1-k)=0$$
as given in the above article.

I found that a graph of the function
$$y=\frac{\theta}{90}\cos 2\theta-\frac{1}{\pi}\sin 2\theta+1$$
as shown in the figure formed a useful basis for an approximate



solution for use in Newton's method. The intersection of the line $y=k$ with this curve will give θ to about $\frac{1}{2}°$.

Using Newton's method on this approximation the following values were then obtained :

k	θ	l
·5	54° 35′ 39″	1·158 729
·333	62° 53′ 1″	·911 597
·25	67° 12′ 31″	·774 754

where l is the length of tie rope per unit field radius.

A. J. Booth.

1562. *An inequality connecting any two triangles.*

In Note 1532 the inequality $a^2+b^2+c^2 \geqslant 4\sqrt{3}\Delta$, which is true for any triangle ABC, was obtained by considering the orthogonal projection of the triangle into an equilateral triangle. It is proved in this note that a triangle ABC can be orthogonally projected into a triangle similar to any given triangle $A'B'C'$, and the following inequality is thence found to hold between the respective sides of the two triangles, and their respective areas Δ, Δ',

$$a^2(-a'^2+b'^2+c'^2)+b^2(a'^2-b'^2+c'^2)+c^2(a'^2+b'^2-c'^2) \geqslant 16\Delta\Delta'.$$

Taking ABC as triangle of reference, consider the inscribed conic whose equation in areal coordinates is

$$p^2x^2+q^2y^2+r^2z^2-2pqxy-2qryz-2rpxz=0.$$

If p, q and r are all positive this conic is necessarily an ellipse ; for the discriminant of the quadratic in $y:z$ obtained by eliminating x between the equation to the conic and the equation $x+y+z=0$ is proportional to $-pqr(p+q+r)$.

Project this ellipse orthogonally into a circle, and suppose that ABC projects into $A_1B_1C_1$. Then the centre of the ellipse, which has coordinates $q+r : r+p : p+q$, projects into the centre of the incircle of $A_1B_1C_1$, and its areal coordinates being unchanged by orthogonal projection we have

$$\frac{q+r}{a'}=\frac{r+p}{b'}=\frac{p+q}{c'}.$$

We need only choose $p:q:r=-a'+b'+c' : a'-b'+c' : a'+b'-c'$ for the triangle ABC to project into a triangle $A_1B_1C_1$ with sides proportional to a', b' and c'.

This having been done, let the sides of $A_1B_1C_1$ be $\lambda a'$, $\lambda b'$, $\lambda c'$. If the heights of the vertices of ABC above the plane of projection

are y, $y+y_1$, and $y+y_2$ respectively, we have the equations

$$a^2=\lambda^2a'^2+(y_1-y_2)^2,$$
$$b^2=\lambda^2b'^2+y_2^2,$$
$$c^2=\lambda^2c'^2+y_1^2.$$

Elimination of y_1, y_2 from these equations yields the equation in λ,

$$16\Delta'^2\lambda^4-2\Theta\lambda^2+16\Delta^2=0,$$

where Δ is the area of ABC, Δ' that of a triangle $A'B'C'$ with sides a', b' and c', and Θ stands for the expression

$$a^2(-a'^2+b'^2+c'^2)+b^2(a'^2-b'^2+c'^2)+c^2(+a'^2+b'^2-c'^2)$$
$$\equiv a'^2(-a^2+b^2+c^2)+b'^2(a^2-b^2+c^2)+c'^2(a^2+b^2-c^2).$$

Since the orthogonal projection is a real one, the above quadratic in λ^2 has real, positive roots. We therefore have $\Theta^2\geqslant(16\Delta\Delta')^2$, and also $\Theta>0$. Therefore $\Theta\geqslant16\Delta\Delta'$, *i.e.*

$$a^2(-a'^2+b'^2+c'^2)+b^2(+a'^2-b'^2+c'^2)+c^2(+a'^2+b'^2-c'^2)\geqslant16\Delta\Delta'.$$

To prove this inequality directly we notice that the expression on the left can be written as

$$2(b^2c'^2+b'^2c^2)-(-a^2+b^2+c^2)(-a'^2+b'^2+c'^2).$$

This is equal to

$$2(b^2c'^2+b'^2c^2-2bc\,b'c'\cos A\cos A')\geqslant4bc\,b'c'\sin A\sin A'=16\Delta\Delta',$$

with equality only if the triangles are similar. I am indebted to Professor Neville for this elegant proof.

The question that still remains is, "What geometrical meaning can be attached to the inequality?" D. PEDOE.

1563. *The cubic equation.*

There have been many articles and notes in the *Gazette* dealing with the nature of the roots of the cubic, but I do not remember having seen the following noted.

Taking the cubic in the form

$$x^3-ax+b=0, \quad \text{.....(i)}$$

when three real roots may be possible for certain values of b, the solutions will be given by the abscissae of the intersections of the curve

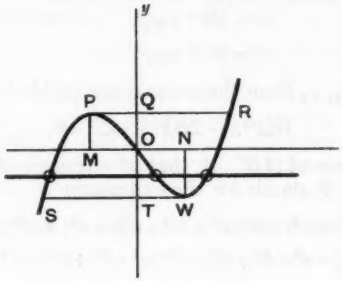
$$y=x^3-ax$$

with

$$y=-b$$

as shown in the diagram.

Now the sum of the roots of the cubic (i) is zero, and hence if two of them are equal and of value α, the remaining root will be -2α.



Referring to the figure, this means that

$$2PQ = QR, \qquad ST = 2TW.$$

Now the product of the roots is $-b$, and hence

$$-2\alpha^3 = -b;$$

also the sum of the products two at a time is $-a$. Thus

$$-3\alpha^2 = -a,$$

whence, in the case of equal roots,

$$4a^3 = 27b^2.$$

Hence if $4a^3 > 27b^2$, there will be three real roots.

Again, referring to the figure, since the curve $y = x^3 - ax$ crosses the axis of x at the points $(\pm\sqrt{a}, 0)$, and since

$$ST = 2TW = 2\sqrt{(\tfrac{1}{3}a)}$$

and

$$2PQ = QR \quad = 2\sqrt{(\tfrac{1}{3}a)},$$

we see that:

(i) if $b > 0$, so that the line $y = -b$ lies below the axis of x, the three real roots must lie within the following intervals of x:

$$\{-2\sqrt{(\tfrac{1}{3}a)}, \ -\sqrt{a}\}; \quad \{0, \sqrt{(\tfrac{1}{3}a)}\}; \quad \{\sqrt{(\tfrac{1}{3}a)}, \sqrt{a}\};$$

(ii) if $b < 0$, so that the line $y = -b$ lies above the axis of x, the roots lie within the following intervals of x:

$$\{-\sqrt{a}, \ -\sqrt{(\tfrac{1}{3}a)}\}; \quad \{-\sqrt{(\tfrac{1}{3}a)}, 0\}; \quad \{\sqrt{a}, 2\sqrt{(\tfrac{1}{3}a)}\}.$$

H. N. HASKELL.

1564. *On Note* 1502.

In the following notes the estimation of $\int_0^a x^x\,dx$, the integral mentioned by Mr. Brewster, has been considered in three stages, namely, $0 < a \leqslant 1$, $a > 3$, and finally $1 < a \leqslant 3$.

(1) It appears that, in the first stage, a convergent series is obtainable, the evaluation of which presents no serious difficulties. It reduces to a known series when $a=1$.

(2) In the case of a large it is found that $\int_b^a x^x\,dx$ can be expanded in the form of an infinite series which unfortunately is divergent. This difficulty, however, can be overcome by terminating the series at any desired stage and obtaining a residual integral which, as it is small, can easily be evaluated with sufficient accuracy. The lower limit has to be set a little above 1, and it is convenient to make $b=3$ or 4.

(3) This leaves a small range 1 to 3 to be evaluated by other means.

The method employed in case (2) is one which I have found exceedingly useful in obtaining solutions of differential equations in series of any required form. It can also be used to obtain the series required for case (1).

(1) a lies between 0 and 1.

Let $u=x^x$ and $v=1+\log x$, then $du/dx=uv$.

By the rule for integration by parts, we have

$$\begin{aligned}\int u\,dx &= ux-\int uv\,.\,x\,dx\\ &= ux-\int u\,.\,vx\,dx,\end{aligned}$$

and since $\int vx\,dx=\tfrac{1}{2}x^2(v-\tfrac{1}{2})$,

we have $\int u\,.\,vx\,.\,dx=u\,.\,\tfrac{1}{2}x^2\,.\,(v-\tfrac{1}{2})-\int uv\,.\,\tfrac{1}{2}x^2(v-\tfrac{1}{2})\,dx.$

Writing this in the form

$$\int u\,.\,xf_1(v)\,dx=u\,.\,\tfrac{1}{2}x^2\,.\,\phi_1(v)-\int u\,.\,\tfrac{1}{2}x^2\,.\,f_2(v)\,dx,$$

it follows that in general

$$\int u\,.\,\frac{x^n}{n!}f_n(v)\,dx=u\,.\,\frac{x^{n+1}}{(n+1)!}\phi_n(v)-\int u\,.\,\frac{x^{n+1}}{(n+1)!}f_{n+1}(v)\,dx,$$

where f_n and ϕ_n are polynomials of the nth degree in v.

From this, $$f_{n+1}(v)=v\phi_n(v)$$

and $$\begin{aligned}\frac{x^n}{n!}f_n(v)&=\frac{d}{dx}\left\{\frac{x^{n+1}}{(n+1)!}\phi_n(v)\right\}\\ &=\frac{x^n}{n!}\phi_n(v)+\frac{x^{n+1}}{(n+1)!}\frac{d\phi_n}{dv}\cdot\frac{1}{x},\end{aligned}$$

or $$\begin{aligned}\phi_n+\frac{1}{n+1}\cdot\frac{d\phi_n}{dv}&=f_n\\ &=v\phi_{n-1}.\end{aligned}$$

That is $$\left(1+\frac{D}{n+1}\right)\phi_n=v\phi_{n-1},$$

and
$$\phi_n = \left(1+\frac{D}{n+1}\right)^{-1} v\phi_{n-1}$$
$$= \left\{1-\frac{D}{n+1}+\left(\frac{D}{n+1}\right)^2-\dots\right\} v\phi_{n-1}.$$

Thus, finally,
$$\int u\,dx = ux\phi_0 - u\cdot\frac{x^2}{2!}\cdot\phi_1 + u\cdot\frac{x^3}{3!}\phi_2 - \dots,$$

where
$$\phi_0 = 1,$$
$$\phi_1 = v - \tfrac{1}{2},$$
$$\phi_2 = v^2 - \tfrac{7}{6}v + \tfrac{7}{15},$$
$$\phi_3 = v^3 - \tfrac{23}{12}v^2 + \tfrac{97}{72}v - \tfrac{97}{255},$$
$$\phi_4 = v^4 - \tfrac{163}{60}v^3 + \tfrac{5359}{1800}v^2 - \tfrac{54997}{36000}v + \tfrac{54997}{180000}.$$

x	u	v	ϕ_1	ϕ_2	ϕ_3	I
0·1	0·7943	−1·30259	−1·8026	3·6054	−7·556	0·0871
0·2	0·7248	−0·60944	−1·1094	1·4713	−2·096	0·1626
0·3	0·6968	−0·20398	−0·7040	0·6685	−0·700	0·2334
0·4	0·6931	+0·08370	−0·4163	0·2983	−0·237	0·3023
0·5	0·7071	0·30685	−0·19315	0·1251	−0·075	0·3726
0·6	0·7360	0·48917	−0·01083	0·0575	−0·019	0·4446
0·8	0·8365	0·77685	0·27685	0·0861	+0·022	0·6009
1·0	1·0000	1·00000	$\frac{1}{2}$	$\frac{2}{9}$	$\frac{3}{32}$	0·7834

In this calculation only the first four terms of the series have been used, and a check on the accuracy can be made by the formula
$$\int u\,dx = u\left(x\phi_0 - \frac{x^2}{2!}\phi_1 + \frac{x^3}{3!}\phi_2 - \frac{x^4}{4!}\phi_3\right) + \int u\frac{x^4}{4!}v\phi_3\,dx.$$

(2) If a is large, let $\int u\,dx = u(f_0+f_1+f_2+\dots)$,
then differentiating $u = uv(f_0+f_1+f_2+\dots)$
$$+u(f_0'+f_1'+f_2'+\dots), \text{ where } f_n' = \frac{d}{dx}f_n,$$
or
$$1 = v(f_0+f_1+f_2+\dots)+f_0'+f_1'+f_2'+\dots.$$
This may be satisfied by taking
$$1 = vf_0, \quad \text{or} \quad f_0 = \frac{1}{v}, \quad f_0 = 1/v;$$
$$0 = vf_1+f_0', \qquad f_1 = -f_0'/v = 1/v^2x;$$
$$0 = vf_2+f_1', \qquad f_2 = -f_1'/v = (3+v)/v^5x^2;$$
$$0 = vf_3+f_2', \qquad f_3 = -f_2'/v = (15+10v+2v^2)/v^7x^3.$$

As the resulting series is divergent it is necessary to obtain a residual integral by taking

$$0 = dI_4/dx + uf_3',$$

or

$$I_4 = \int \{u(105 + 105v + 40v^2 + 6v^3)/v^3x^4\}\, dx.$$

x	u	v	f_0	f_1	f_2	f_3	$u\Sigma f$
3	27	2·0986	·476	·036	·014	·009	14·4
5	3125	2·6094	·38597	·01134	·00187	·00054	1249·1
10	10^{10}	3·3026	·30280	·00278	·00016	·00002	$3{\cdot}0576 \times 10^9$
20	$1{\cdot}0486 \times 10^{26}$	3·9957	·25100	·00079	·00002	—	$2{\cdot}6405 \times 10^{25}$

Using Simpson's rule with two strips the value of the residual integral between $x = 3$ and 5 is found to be 1·4, while between $x = 3$ and 10 it is negligible.

Thus $$\int_3^5 u\, dx = 1249{\cdot}1 - 14{\cdot}4 + 1{\cdot}4 = \underline{1236},$$

and $$\int_3^{10} u\, dx = 3{\cdot}0576 \times 10^9 - 14{\cdot}4 = \underline{3{\cdot}058 \times 10^9},$$

and $$\int_3^{20} u\, dx = \underline{2{\cdot}6405 \times 10^{25}}.$$

In fact, if $a \geqslant 6$, both the residual integral and the lower limit can be ignored, and we have the formula

$$\int_0^a x^2\, dx = \frac{u}{v}\left(1 + \frac{1}{v^2x} + \frac{3+v}{v^4x^2} + \frac{15 + 10v + 2v^2}{v^6x^3}\right).$$

(3) Within the region $x = 1$ to 3 the expansion used in (2) is too rapidly divergent. The integral can be evaluated directly by means of Simpson's rule and the result obtained, using 10 strips, is 13·728, but this is not reliable on account of the rapid variation of u.

An interesting transformation can be obtained from the equation

$$\frac{du}{dx} = uv,$$

$$u = \frac{1}{v}\frac{du}{dx}.$$

whence $\int u\, dx = \int (1/v)\, du = \int w\, du$, where $w = 1/v$.

The graph of this function could be used for rapid approximate evaluation over a given range.

x -	1·0	1·25	1·5	1·75	2·0	2·25	2·5	2·75	3·0
u -	1·000	1·322	1·837	2·663	4	6·200	9·882	16·150	27
w -	1·000	0·817	0·712	0·641	0·592	0·552	0·522	0·497	0·476

Simpson's rule cannot be used as it is not possible to calculate values of w for equidistant values of u. It is possible, however, to write $\int_b^a w\, du = \Big[wu\Big]_b^a - \int_b^a u\, dw$, and to calculate values of u for equidistant values of w over any required range. The method is

rather clumsy, but as the main part of the result is the first term $\left[wu\right]_b^a$ it gives an accurate result. If $b=1$ and $a=3$, we have

$$\int_1^3 u\,dx = (u_3w_3 - u_1w_1) + \int_1^3 u\,dw$$
$$= (12{\cdot}866 - 1) + 1{\cdot}888$$
$$= 13{\cdot}754,$$

giving $$\int_0^3 u\,dx = 13{\cdot}754 + 0{\cdot}783 = \underline{14{\cdot}537}.$$

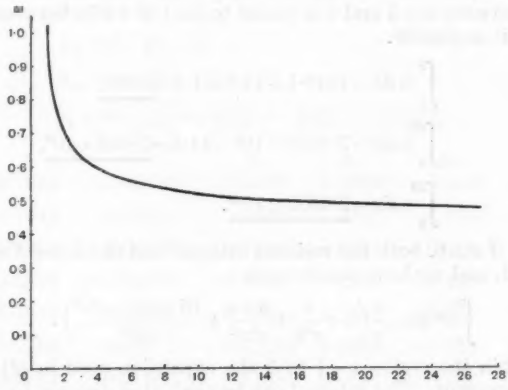


FIG.

Finally, summarising the above results :

a	0·1	0·2	0·4	0·6	0·8	1·0	3	5	10	20
I	0·0871	·1626	·3023	·4446	·6009	·7834	14·537	1251	$3{\cdot}058\times10^{9}$	$2{\cdot}641\times10^{55}$

W. HOWARD JOINT.

1565. *Use of the fractional method in long multiplication of money.*

In the *Mathematical Gazette* for May, 1941, Mr. Webb suggests that further experiments be made to compare his method of long multiplication with that of Mr. Siddons. I would prefer that the possibility of improving Mr. Webb's method, by utilising the fractional method of multiplying pence, be first considered. In Mr. Siddons' method it is impossible, in finding the partial products, to use economical devices for multiplying aliquot parts of one shilling, *e.g.* $6d.\times249$ must be worked out as $((6\times249)\div12)$s. and not as $(249\div2)$s. In Mr. Webb's method these devices could be used without complicating unduly the setting out, but apparently they are deliberately excluded. I suppose this is done in order to avoid variety in the working and so ensure the more complete mechanisation of the whole process. I am doubtful whether it

results in the maximun speed and accuracy. In multiplying 2d., 3d., 4d., or 6d., by the fractional method there is much less working than in ordinary multiplication. With ¼d., ½d. or ¾d. also in the multiplicand the matter needs investigation. Using Mr. Webb's method I have shown below 1s. 3½d. × 539 in two ways. In (1) 3d. is multiplied by ordinary multiplication, and in (2) by the fractional method. The multiplier has been chosen to give the least favourable case for the fractional method.

(1)

1s.	539 3½d.
696s.	2½d.
157	269
539	1617*A*
696*D*	12) 1886*B*
	157*C*

(2)

1s.	539 ¼s. ½d.
696s.	2½d.
23	269
134	9*A*
539	12) 278*B*
696*D*	23*C*

An analysis for each sum is given below. It shows the number of operations involved in each step, *A*, *B*, *C*, *D*, and the arithmetical nature of the operations. Adding a carrying figure, in either addition, subtraction or multiplication, counts as one operation. So too, in division, does each partial quotient, the remainder (unless 0) for each partial quotient, and translating a final remainder into pence.

(1)	Add.	Sub.	Mult.	Div.
A	2		3	
B	4			
C		3		3
D	4			
Total	10	3	3	3

(2)	Add.	Sub.	Mult.	Div.	Rem. to d.
A		3		3	1
B	2				
C		2		2	
D	6				
Total	8	5		5	1

The total number of figures written down in operations *A*, *B*, *C*, *D*, is 22 in (1), and 19 in (2).

The grand total of operations is the same, 19, in each sum. As each grand total contains, at least, 8 additions, and 2 divisions by 12 (with 2 corresponding subtractions), I have subtracted all these from the corresponding totals so as to show clearly the difference between (1) and (2), thus :

	Add.	Sub.	Mult.	Div.	Rem. to d.
(1)	2	1	3	1	
(2)		3		3	1

There seems little to choose between (1) and (2). The division, with subtraction, is harder in (1) than those in (2). But multiplication is generally regarded as easier than division, while turning a fractional remainder into pence may be difficult. Much depends on the extent to which the different processes are routinised. This point is discussed below.

If the multiplier is exactly divisible by 4 the results are, of course, more favourable to the fractional method. Step B and part of A disappear. The results for a typical case, 1s. $3\frac{1}{2}$d. $\times 536$, in shortened form, are as follows :

	A	B	C	D	Add.	Sub.	Mult.	Div.	Total
(1)	5	4	6	4	10	3	3	3	19
(2)	5	0	4	6	6	4	0	5	15

Total number of figures in A, B, C, D ; 22 in (1) ; 15 in (2). The odds that a case of this kind may occur vary from 1 in 2 for 6d. to 1 in 6 for 2d. These are high enough to make the total possible advantage appreciable.

There is one other factor which may operate in favour of the fractional method. The multiplier may be so low that the number of pence to be brought to shillings in (2) may come within range of the shillings-pence table which the pupil has committed to memory. He then does the reduction automatically and a formal division sum is unnecessary. This does not affect the number of operations under my scheme of assessment, but it must result in a definite saving of time, and it does reduce the amount of writing. Assuming that the table up to 120d. = 10s. is known, these advantages would be secured in multiplying by any number from 160, for $\frac{3}{4}$d. in the multiplicand, to 480, for $\frac{1}{4}$d. This is an important consideration, for most pupils, and particularly those in Junior and Senior schools, are unlikely to meet with larger multipliers.

On the whole the facts suggest that the fractional method should be used in multiplying any sum of money containing 2d., 3d., 4d., 6d., plus 0d., $\frac{1}{4}$d., $\frac{1}{2}$d., $\frac{3}{4}$d., in the pence column, provided the arithmetical processes involved can be performed as well as those of ordinary multiplication. The pupil must be able to divide as well as he multiplies, and turn certain fractions of a shilling to pence as well as he divides by 12. I see no reason why he should not do so, if he is taught with this end in view. A fundamental change of outlook is needed. At present the fractional method is generally regarded merely as an economical alternative to the regular method of multiplying. It embodies a few devices suitable for specific occasions. The good pupil may note a suitable occasion, and may use an appropriate device if he wish. For other pupils routine procedure is safer. The result of this attitude is that the fractional method is neither thoroughly taught nor systematically used. I would suggest that the teacher recognise it as the essential method of multiplying certain aliquot parts of 1s., and that he insist on its use in these cases, instead of ordinary multiplication.

The British monetary system, based on the numbers 12 and 20, is often compared adversely with the decimal system. But these numbers, with their great divisibility, make the British units admirably adapted to fractional treatment. This is not accidental, for the British system was, in fact, historically evolved in connection with the simple unit fractions, as our coins, including the obsolete groat and the rare four-shilling piece, illustrate. It would therefore seem natural that the pupil should not only study Money and Fractions in close connection, but also use the simple fractions in money calculations. When he can deal with halfpence and farthings let him study 6d. as $\frac{1}{2}$s. and 3d. as $\frac{1}{4}$s. The terms "half-shilling" and "quarter-shilling" could be used. He would learn that to multiply 6d. by any number involves exactly the same process as multiplying $\frac{1}{2}$d. It is at this stage that I suggest the old method of multiplying 6d. be entirely abandoned. It has served its purpose and may be replaced as completely as algebraic methods yield to those of the calculus at a later stage. The pupil would apply other unit fractions similarly, as he made acquaintance with them on the ruler or elsewhere, and use them as essential instruments for multiplying money. Whether the teacher decided to treat only $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, or $\frac{1}{6}$ and $\frac{1}{8}$ also in this way, the work would be standardised, and the pupil would learn to set down the remainder, in dividing, automatically, expressing it either as pence or a fraction of a shilling with equal facility. This would entail memorising only eleven facts of the type $\frac{3}{4}$s. = 9d., to use in multiplying 2d., 3d., 4d., 6d. The use of non-unit fractions in long multiplication would on the whole give no advantage except in multiplying 4$\frac{1}{2}$d. and 7$\frac{1}{2}$d. as $\frac{3}{8}$s. and $\frac{5}{8}$s. 10$\frac{1}{2}$d. offers a decided advantage, but only if treated subtractively as 1s. $-\frac{1}{8}$s. The subtractive method applied to 1s. 11d. plus 0d., $\frac{1}{4}$d., $\frac{1}{2}$d., $\frac{3}{4}$d. gives similar advantages, and it entails only a slight adjustment of the form of setting out. But the use of any procedure derived from either the fractional or the subtractive method must depend on the pupil's ability to apply it mechanically. It is my belief that all pupils could use some of the unit fractions, if only $\frac{1}{2}$ and $\frac{1}{4}$, as the regular means of multiplying. Experiments would show how far this belief is valid, but they must be based on a proper reorientation of the teaching of Money and Fractions.

R. S. WILLIAMSON.

1382. But the truth is, that the knowledge of external nature, and the sciences which that knowledge requires or includes, are not the great or the frequent business of the human mind. . . . Prudence and Justice are virtues and excellencies of all times and of all places; we are perpetually moralists, but we are geometricians only by chance.—Johnson, *Lives of the Poets*, s.v. Milton, 1779. [Per Professor E. H. Neville.]

1383. "I am a mathematician, sir. I never permit myself to think."—J. D. Carr, *The Hollow Man*, p. 40 (1935). [Presumably Newton's "Hypotheses non fingo" less happily expressed.] [Per Professor E. H. Neville.]

REVIEWS.

The theory and applications of harmonic integrals. By W. V. D. HODGE. Pp. ix, 281. 21s. 1941. (Cambridge)

The author and publishers of this compact volume are surely to be congratulated on its bearing the date 1941. There is no sign of hurry or distraction in its editing; a complete list of errata would be very short in spite of the fact that the printed formulae are often complicated, with suffixes and sub-suffixes throughout.

It is about a century since Riemann wrote on algebraic integrals, founding his method on the theory of potential or harmonic functions of two real variables. It must have occurred to every student in this field since his day that an analogous method for many variables would yield fruitful results, but, although many wide extensions have been made, Professor Hodge has been the first to give substantial meaning to harmonic functions in general and to apply them to the study of integrals of complex variables on an algebraic variety. This is therefore a remarkable book and much may be expected from the development of the devices which it puts in the hands of future students. The exposition is lucid, straightforward and free from pedantry, a freedom which is especially noticeable in the choice of notation. The terseness of the style does not make the book more difficult to read, but a more discursive paragraph occasionally might have led readers to a higher appreciation of the results. How much can be done in this direction is seen by looking at the opening paragraph of each chapter, which gives an admirable sketch of the purpose of the chapter. The bibliography is strictly relevant and little reference is made to earlier work. The relation between the ideas of a Riemann surface and a Riemann manifold is discussed without mention of Klein, the integrals on a manifold are the main subject of the book and yet Poincaré is not referred to, while the chapter on continuous groups is almost free from the name of Lie. Thus the book is essentially forward-looking and the reader is accepted with a background already filled in.

The essential chapters of the book are the second and third, which deal with integrals on an orientable manifold and harmonic integrals on a manifold with a metric. A manifold is a connected system of neighbourhoods and admissible coordinate systems, and is most comfortably dealt with by considering any one of its covering complexes. The study of an integral on the complex and the manifold consists of the examination of a differential form (i) locally, *i.e.* in a neighbourhood and immediately adjoining neighbourhoods, and (ii) with respect to the path or chain of integration over which the differential form is continued; (i) requires the use of tensor calculus, and (ii) involves a knowledge of the topology of the complex and manifold as a whole. In the introductory chapter is given a brief résumé of these two theories. The account of tensor theory is clear and adequate in itself, and it may be said that in all the somewhat laborious tensor work which follows there are none of those gaps (except when due warning is given) which are apt to make such work unreadable as well as tedious. The section on combinatory topology is more exacting but well fulfils its purpose of delimiting the equipment required of a reader. Attention is confined to the Riemann manifold though no attempt is made to simplify expression by intuitive notions or by taking advantage of the metrisability of the manifold. The intention is rather to facilitate reference by following the text-books on the subject as closely as possible. The main variation arises from the use of the notion of *class*, familiar to readers of Veblen and Whitehead's tract: if the functions introduced in the defining of a complex, manifold or correspondence have continuous derivatives of order u,

it is said to be of class u. This number is required for any precise statement about the integrals and it is necessary to see in what sense the topological statements are independent of it. (But according to the definition things of class $u+1$ are also of class u and the oft-repeated phrase " class u at least ", though well-intentioned, is misleading.) Paragraphs on the dual complex, the intersection of cycles, and the product manifold provide the machinery used systematically throughout the book.

It is interesting to observe how the notation and ideas of topology extend themselves to differential p-forms (linear forms in p-fold differentials). The formula of Stokes when generalised is $\int_\beta A = \int_\alpha B$, where β is the boundary of the p-chain α (of p dimensions) on the manifold and B is the exterior derived p-form of the form A. It manifestly expresses a contragredience between the forms and the chains, in which the derivative of the form corresponds to the boundary of the chain, and this sort of duality is exploited in the notation and in the proofs where it is replaced or balanced by the introduction of the dual complex of Poincaré. Thus $\beta \sim 0$ means that β is the boundary of a chain; $B \sim 0$ that B is a derived form. When α is closed (a cycle) we write $\alpha \to 0$; when the derived form of A vanishes, we write $A \to 0$ and say that A is *closed* (instead of *exact*). Then just as $\alpha \sim 0$ implies $\alpha \to 0$, so $A \sim 0$ implies $A \to 0$. Moreover if $A \to 0$ and $\alpha \sim 0$, or if $A \sim 0$ and $\alpha \to 0$, then $\int_\alpha A = 0$. There are converse theorems, one of which was badly wanted for some years before it was proved by de Rham in 1931, viz., if $A \to 0$ and $\int_\alpha A = 0$ for every $\alpha \to 0$, then $A \sim 0$. Whence it follows that the number of linearly independent (for $\sim$) closed p-forms on a manifold is the pth Betti number; and as it is a principal property of a harmonic form that $H \sim 0$ implies $H = 0$, this Betti number is the number of linearly independent (for $=$) harmonic p-forms on the manifold. The crux, however, of proof lies in the construction on any given Riemann manifold of a closed A, or of a harmonic H, having assigned periods. Professor Hodge reproduces in more explicit form his second method of solving this fundamental problem; the construction is made to depend on the solution of an integral equation.

A reviewer in this periodical is absolved from the attempt to give a précis of the theory of harmonic forms as applied to an algebraic variety by the fact that an excellent summary in simple terms has been contributed to the *Gazette* * by the author himself. As it appears in this book the main object is to find a suitable base for the harmonic forms on the Riemann manifold associated with the variety. The metric chosen for the manifold is the simplest possible, but for this simplicity a heavy price is paid in that the metric, and consequently the harmonic forms, are not invariant under birational transformation of the variety. The process is therefore artificial in a sense that is clearly explained. Yet the linear base found for the harmonic forms is a series of sets which correspond set for set with the base for cycles on the manifold as extracted from the previous work of Lefschetz. Each set of harmonic forms has periods only on the cycles of the corresponding set. These periods form a non-singular matrix, and the period-matrices are birational invariants of the variety. Further, members of each of the sets are segregated formally into familiar types of complex forms and the period-matrix of each type is a birational invariant. The simplest of these types are the algebraic integrals of the first kind, and some of the results refer mainly to them. But

* XVII, p. 303, December 1933.

the novelty of the method appears in the use of the different types and their interplay, culminating in a remarkable extension of Poincaré's theorem on reducible integrals. The last paragraphs of the chapter, which are addressed to algebraic geometers, show that striking results in their subject have already been obtained by these transcendental methods.

The last chapter deals with the applications of harmonic forms to the theory of continuous groups. A clear account is given of this theory as far as it is required, but the object of the chapter is to throw light upon harmonic integrals; readers are referred to the original papers of Cartan, which should certainly be consulted if only to mitigate the severely analytical treatment presented here. The interest in the application is two-fold; firstly, in contrast with the preceding chapter, the metric chosen on the manifold is intrinsic to the problem—the harmonic integrals themselves are invariant, and secondly, the initial conception of the topological manifold itself is different.

The manifold is defined by a continuous group of transformations which change one point of it into another. More precisely it is defined by the continuous group G which thus operates on its points and a sub-group g of G which leaves one particular point, O, fixed. Invariant integrals have been used in groups to provide a means of measurement; as their name implies, they are unaltered under any transformation of the group. The invariant volume used by Weyl is a particular example which appears to be sufficient in his hands for most of the problems which have been solved, but it is natural to extend the notion to p-integrals which elegantly replace his algebraic invariants. For the integrals to be proper, the group must be compact (or closed), *i.e.* any infinite set of operations of the group must have a limit inside the group.

Suppose for simplicity that G is a simply transitive group (g=identity). Then the invariant forms can be written down at once as the sum of constant multiples of the standard exterior differential products. A group G_1 with $g=(1, T)$, where $T^s=1$, may be regarded as an extension of G with a consequent tightening up of the manifold which is now symmetrical about the point O, and consequently about every one of its points. Here the invariant forms are as written down before but conditioned by invariance under an involution, which implies certain linear relations between its coefficients and the constants of structure of the group. It is remarkable that these are precisely the conditions that the invariant form should be closed.

The only compact groups dealt with in this book are simple groups or groups which are the direct product of simple groups. For any such group an intrinsic metric is introduced. The manifold is then a Riemann manifold (actually an Einstein space) and there are harmonic forms on it. The simply transitive group G is now the group of translations of the manifold into itself, while in the extended case (G_1) there exists a symmetry on the manifold whereby patches symmetrically placed with respect to a point are isometric. On either of these manifolds the harmonic forms are invariant forms, since a closed form is carried over into a closed form and moreover into a homologous form. But in the extended case, since all invariant forms are closed, the harmonic forms are the *only* invariant forms; hence the linear conditions for the invariant forms determine the harmonic forms and in particular the Betti numbers of the manifold.

For the general case of a transitive group the sub-group of rotations of the manifold into itself has also to be considered. On the other hand the group of parameters of any such group is simply transitive and, when extended, defines the symmetrical "group manifold", on which it appears as a group of translations. The last paragraphs of the chapter are devoted to the detailed

investigation of the harmonic forms and Betti numbers of the group manifold in the case of each of the four main classes of simple groups. These make a fitting epilogue to a beautifully arranged book. P. F.

An Introduction to Differential Geometry with use of the Tensor Calculus. By L. P. EISENHART. Pp. x, 304. $ 3.50. 1941. Princeton Mathematical Series, No. 3. (Princeton University Press; Oxford University Press.)

Professor Eisenhart's earlier treatise on differential geometry has for many years been a standard work. In the volume under review, which covers to a large extent the same ground, the Cartesian methods employed in the earlier book are replaced by the systematic use of the tensor calculus, the foundations of which are clearly explained.

The first chapter deals with the theory of space curves: here the treatment follows the usual lines and employs Cartesian methods. The second chapter introduces the concepts and technique of the tensor calculus, and the rest of the book consists of applications to the differential geometry of surfaces in ordinary space, considering first the intrinsic geometry on the surface (geodesics, geodesic curvature) and subsequently the properties of the surface in relation to external space (lines of curvature, asymptotic lines, etc.).

The tensor calculus has come into prominence largely because of its eminent suitability for handling problems of differential geometry in higher space; a suitability due in the main to the fact that it deals throughout with invariant concepts, but in part also to the fact that its notation is so contrived that very complicated formal expressions can be expressed in a very concise form and manipulated conveniently. In the relatively simple situations which arise in elementary differential geometry, the gain in formal conciseness obtained by using this calculus in preference to—say—the methods of ordinary vector algebra is not large, as a comparison of this book with texts using vector methods will show. The advantages of the tensor methods are, firstly, that they lend themselves more freely to generalisations of the theory to space of higher dimensions, and secondly that they lead inevitably to a clear distinction between the properties intrinsic to the variety considered and these properties which involve its relations to the space in which it lies. The value of a work such as this lies in the fact that the important concepts of the tensor calculus are introduced at an early stage, and the reader who has mastered them will be adequately equipped to follow the generalisations of the theory such as Riemannian and non-Riemannian geometries.

As we should expect from its author, the exposition of the subject matter is very clear, and the book should become a standard text; it will certainly be on the bookshelf of anyone who has to teach the subject, and ought to be read by those whom they teach. A feature worthy of special mention is the excellence of the illustrations. There is a useful collection of exercises following the various sections, a short bibliography and an index. The book is attractively printed, and ought to find a large circle of readers.

J. A. TODD.

General Analysis, II. By E. H. MOORE. Pp. vi, 255. 14s. 1939. (American Philosophical Society, Philadelphia; Humphrey Milford)

The book before us is the second of a series of four volumes devoted to an account of E. H. Moore's General Analysis. A brief outline of this theory will be found in the review of part I (*Math. Gazette*, XX (1936), 215). Part I was mainly concerned with matrices and determinants of finite order whose elements belong to some number field in which multiplication is not necessarily commutative. In part II a far-going generalisation of the idea of limits is

introduced and an extensive study is made of a number of important notions connected with it. Axiomatic reasons, notably the validity of some equivalent of Dedekind's continuity axiom, restrict the underlying number field $\mathfrak{A}$ to one of the following systems: (i) all real numbers, (ii) all complex numbers, (iii) all quaternions with real components. In $\mathfrak{A}$ every number has a conjugate $\bar{a}$, and $|a| = (a\bar{a})^{\frac{1}{2}}$ is an ordinary non-negative number. Numbers a satisfying $\bar{a} = a$ are ordinary real numbers.

In 1922 E. H. Moore and H. L. Smith developed an extension of the definition of ordinary limits $\lim_{l \to \infty} a_l$ which comprises as special cases many generalised limits used by mathematicians. They start from an abstract class $\mathfrak{L}$ of objects l, of finite, denumerable or, indeed, any arbitrary power. (In the case of ordinary limits $\mathfrak{L} = \{1, 2, 3, \ldots\}$.) Between certain ordered pairs l_1, l_2 of equal or distinct elements of $\mathfrak{L}$ a relation R is defined, $l_1 R l_2$ being the notation for the fact that l_1 stands to l_2 in the relation R. (In the case of ordinary limits $l_1 R l_2$ means $l_1 > l_2$.) R is assumed to have the following properties: (i) if $l_1 R l_2$ and $l_2 R l_3$, then $l_1 R l_3$; (ii) given any l_1, l_2 of $\mathfrak{L}$ there is at least one element l of $\mathfrak{L}$ satisfying both $l R l_1$ and $l R l_2$.

With every l a number a_l of $\mathfrak{A}$ is associated. Then a number a is called the generalised limit of a_l relative to $\mathfrak{L}$ and R, in symbols, $a = \mathrm{L}_{\mathfrak{L}R}\, a_l$ if, given any positive number ϵ, there exists an element l_ϵ of $\mathfrak{L}$ such that $|a_l - a| \leqslant \epsilon$ whenever $l R l_\epsilon$.

If, in particular, $\mathfrak{L}$ is the set of all finite sub-sets of a given set $\mathfrak{N}$ (elements n), and if $l_1 R l_2$ means that l_1 contains l_2 as a sub-set then a definition is obtained of convergent generalised sums $\sum_{n \text{ in } \mathfrak{N}} b_n$. Simply put

$$s_l = \sum_{n \text{ in } l} b_n; \qquad \sum_{n \text{ in } \mathfrak{N}} b_n = \mathrm{L}_{\mathfrak{L}R}\, s_l = s.$$

Moore calls s the limit of s_l " as l swells in $\mathfrak{N}$ ".

The first chapter of the present volume (chapter IV of the series) is concerned with upper and lower bounds and limits, iterated and double limits and sums and various modes of uniform convergence. Definitions, theorems and proofs are straightforward generalisations of the corresponding parts of ordinary analysis. Let $\xi(p)$ be any " function " or " vector ", defined for all elements p of a given abstract set $\mathfrak{P}$ and having its functional values or components in $\mathfrak{A}$. In chapter V a metric is introduced into the space of all ξ, by means of a " fundamental matrix " $\epsilon(p_1, p_2)$. As was described in the review of part I *, every vector ξ has, with respect to ϵ, a modulus $M\xi$ which is a non-negative number or ∞. Moore's theory is mainly concerned with the sub-space $\mathfrak{M}$ of modular vectors, *i.e.* of vectors whose modulus is finite. When in particular $\mathfrak{P} = \{1, 2, 3, \ldots\}$ and $\epsilon(p_1, p_2)$ is 1 or 0 according to whether p_1 is equal to or differs from p_2, then $\mathfrak{M}$ becomes Hilbert's space of all sequences $(x_1, x_2, \ldots)$ for which $\sum_1^\infty |x_m|^2$ converges. ϵ serves also to define an inner product of two vectors.

The concluding two chapters VI, VII deal with matrices which have any arbitrary number of rows or columns, and also with the linear transformations associated with them. Known theorems on matrices with denumerably many rows or columns are extended. Although definitions and theorems are fairly

* *Loc. cit.*, p. 216. I should like to point out that on the right-hand side of the first formula on p. 216, σ should be replaced by ϵ.

complicated it seems obvious that some of them will be useful tools for the remaining two volumes of this series entitled *General Fourier Series and Modular Spaces*, and *The Characteristic Value Problem in General Analysis* respectively. R. R.

Fourier Series and Boundary Value Problems. By R. V. CHURCHILL. Pp. ix, 206. 17s. 6d. 1941. (McGraw-Hill.)

This book is attractively printed, and the subject matter is presented in a very interesting way. It is intended to form an introduction to Fourier series, and to their application to the solution of boundary value problems, connected with the partial differential equations which occur in physics and engineering. It will be followed shortly by a more advanced book on further methods of solving boundary value problems.

The first object of the book is to introduce the reader to the concept of orthogonal sets of functions, and to the ideas of the use of series of such functions in representing an arbitrary function. Considerable stress is naturally laid on the case of Fourier series, but Fourier integrals are introduced, and chapters are also included on Bessel functions and Legendre functions.

The second object of the book, as the title implies, is to give a thorough acquaintance with the classical process of solving boundary value problems with the aid of expressions in series of orthogonal functions. Not only is the formal method of solution given, but attention is also paid to conditions under which the solution is valid, and also to the question of the uniqueness of the solution.

The book should appeal to students of both pure and applied mathematics. To the former it gives a rigorous introduction to the theory of Fourier series, while to the latter it gives both the formal methods of solution and sound proofs under conditions which should cover most of the practical applications. An idea of the scope of the theoretical part of the book will be given by the conditions under which the Fourier expansion of a function $f(x)$ is proved to be valid. These conditions are that $f(x)$ is sectionally continuous in the interval $(-\pi, \pi)$, that is, the interval can be divided into a finite number of intervals in each of which the function is continuous and has finite limits as the variable approaches either end point from the interior; it is then proved that the Fourier series converges to the usual sum at every point where $f(x)$ has a right-hand and a left-hand derivative. The Fourier integral and the expansions of functions in series of Bessel and Legendre functions are also treated individually, but less fully.

As the author says, the book is intended to be both elementary and mathematically sound. It seems evident that there was room for a treatment of Fourier series which neither attempts to prove the validity of a Fourier expansion under very wide conditions nor emphasises the practical applications without proving anything. The author has filled the gap by producing a book in which he gives a rigorous proof of the Fourier expansion for a function which is (roughly speaking) piecewise continuous and differentiable, and he also gives proofs of such theorems as those on differentiating and integrating a Fourier series, and of Parseval's theorem.

The result is an extremely useful book in which the theory, though rigorous, is as elementary as possible, and it should be very acceptable to the class of students for whom it is intended. Perhaps it should be mentioned that references are given to more advanced works on the subject, so that the reader who wishes to go further in the theory will find it easy to pursue his studies.

W. N. B.

New Method Arithmetics. By J. MURRAY. Pupil's Book III. Pp. 95. 1s. 7d. Limp cloth. Teacher's Book III. Pp. 62. 2s. 9d. 1941. (University of London Press)

Book III * of this series provides a course of work in arithmetic for pupils of nine years of age. The work has been well planned and a great deal of care has been spent in grading the exercises. The examples have been chosen to give adequate practice in every number combination and in every kind of difficulty before passing on to the next. The 0, 7, 8 and 9 number combinations (see p. 17 of the Report of the Metropolitan Training Colleges Association on Arithmetic in Junior Schools) seem to cause the greatest difficulty to pupils. I believe that the reason for the zero difficulty may be that so many schools fail to deal with the zero combinations when taking work on the multiplication tables. I have investigated this subject and found that pupils from schools which dealt with the nought combinations found little difficulty in handling them. Although the author has paid attention to the need for eradicating these ascertained difficulties I should have preferred to have seen a few more practices with 7 and 9, particularly with the 7, which seems to create the greatest number of difficulties. There has also been a careful apportioning of practice to the different factors used in the simple multiplication and division of compound quantities. In the former case the pupils are prepared for subsequent work on long multiplication according to the Ballard Box Method.

The chapters on fractions are well done and copiously illustrated. In addition there are interesting sections on time telling, weights and measures, shopping and periodic exercises in problems of a really practical type.

The teacher's book gives valuable hints and general comments on the subject matter, extra examples for practice and answers to the questions contained in the pupil's book.

My own experience is that this series is one of the best yet produced.

H. WEBB.

First Stage Mathematics, suitable for the R.A.F. and A.T.C. By A. F. BUCHAN and R. BORTHWICK. Pp. 96. 2s. 0d. 1941. (University of London Press.)

This little book is written for the " thousands of young men who wish to join the R.A.F., but who feel that they have long forgotten their school mathematics ". It must be supposed that the authors have in mind only ex-secondary schoolboys, for the ground is covered far too quickly for the tens of thousands of ex-elementary-schoolboys who also wish to join the R.A.F.

The A.T.C. syllabus in mathematics is covered, with the addition of chapters on the compass, logarithms, and trigonometry as far as the sine and cosine rules. The chapter of Arithmetic includes decimals, fractions, percentage, average, simple areas and volumes, square root and " Pythagoras ". The eleven pages of Algebra deal with change of subject of formulae, simple and simultaneous equations, quadratic equations (by formula only), and problems. The Geometry consists of a few simple facts about angles, parallels, and parallelograms, with the construction of triangles and scales. The chapters on graphs and triangle of velocities are clear and concise.

In a book designed as a quick " refresher " course one would expect some attention to the commoner errors, but there is no warning about signs in relation to brackets, or about the confusion of ab^2 and $(ab)^2$. There is no

* Books I and II were reviewed on pp. 300-301 of the *Mathematical Gazette* for October 1940.

suggestion that the fourth figure in an answer obtained by the use of four-figure tables is unreliable. There are lapses from good style :

" $\log N = 2\ )\ \overline{2{\cdot}6780}$ ".
$\quad\quad\quad

table being exact only to four decimal places!" There is a short section on interpolation from tables, but the authors are themselves careless on *extra*-polation. In Ex. 6, p. 21, we are told that the atmospheric pressure falls *approximately* (my italics) one millibar for every 30 ft. rise in altitude, and that the pressure at ground level is 1013·2 mb. We are asked to find the pressure at various heights. For 30,000 ft. the answer given is 13·2 mb. Do the authors believe that above 30,400 ft. the atmospheric pressure becomes negative? And this book is for the use of men who as navigators will have to "pin-point" their targets, or as ground engineers to work to thousandths or even ten-thousandths of an inch!

The sequence of topics in the Algebra section is unusual. The "first four rules" are followed by equations, including the inevitable quadratic with its accompaniment of time-honoured conundrums, "transposition" of formulae (quite hard ones) and lastly the construction of formulae: substitution in practical formulae is absent.

The Geometry is confined almost entirely to navigational work, which is well done throughout. The section on Graphs and Coordinates is useful and practical, but the graphical solution of simultaneous and quadratic equations could have been dispensed with. There are sections on Logarithms and the trigonometry of the right-angled triangle. The ratios are defined for angles up to 360°, but angles beyond 90° are used only in such questions as "What is sin 110° 6′?" All answers are given to four figures without comment. There are twelve pages of tables.

It is, of course, realised that any book written specially for the A.T.C. must have been prepared in a hurry, but that does not excuse such statements as those below, especially as one of the authors, himself responsible for several excellent works, gives his address as "Education Department, Cambridge University Press":

"... the quantity that was squared to give the number." "We can now solve $(x + \text{a number})^2 = \text{a number}$." "The ratio circumference : diameter is a fixed number $\pi = 3{\cdot}14$." "By antilogging." "All quadratic equations containing second powers of x, *i.e.* x^2, must have graphs that cross the axis of x in two places!" "The expression $36 = 36$ used above as an illustration of cross-multiplication is what is termed an 'identity'."

Surely it is of first importance to accustom the practical man to accurate thought and clear expression? A. P. R.

1384. An appeal to the mathematicians of all countries is made by the Greek Mathematics Society, in the name of the birthplace of mathematics, the country of Pythagoras, Plato, Euclid, Archimedes, and Apollonius (says Reuter).

The signatories protest against the "unprovoked and barbarian aggression of modern Italy", referring to the torpedoing of the cruiser *Helle* two and a half months before the declaration of war, and the "savage murder by bombing" of the civilian population of Greek cities.

"We are fighting for freedom, for our altars and our hearths, and for the highest ideals of civilized man (they declare). We shall fight to the end in accordance with the national traditions of our history over the past 3,000 years. We appeal to the mathematicians of all countries to give every help to little Greece, so that the lofty principles of freedom and justice and the noble ideals of truly civilized humanity may prevail."—*Times*, November 6, 1940.

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SYDNEY BRANCH.

Report for 1940.

There are now 120 Associate members of the Sydney Branch and 20 full members: 140 in all.

The first meeting of the year was that of the Organising Committee, at which the general plan for the year's activities was discussed; the examination papers in mathematics for the Intermediate Certificate and Leaving Certificate examinations were reviewed; and nominations made for office-bearers for 1941.

There were three general meetings. At the first of these an address was given by Dr. I. S. Turner on "The place of mathematics in a general education". Dr. Turner reviewed opinions on this question current in Great Britain and the United States, in which countries he had recently studied and travelled. It was pointed out that the philosophy underlying the kind of education provided by any country needs to be studied and understood before one can appreciate what that country is trying to do about the education of its adolescent population.

At the second meeting an address was given by Dr. R. L. Aston on "Aerial photography". The lecture was given in the Engineering School, and was freely illustrated by diagrams and to some extent by actual material used in aerial photography.

The address at the third meeting was given by Mr. A. P. Treweek on "The life and work of Pappus". Mr. Treweek had much of interest to say, as he was able to refer to original sources as well as to published works on Pappus.

The office-bearers for 1941 were elected at this meeting as follows: *President:* Dr. I. S. Turner; *Vice-Presidents:* Mr. H. H. Thorne, Professor E. M. Wellish; *Joint Hon. Secretaries:* Miss E. A. West, Mr. H. J. Meldrum; *Hon. Treasurer:* Mr. R. J. Gillings; *Director of Problem Bureau:* Mr. R. J. Gillings.

H. J. Meldrum, Hon. Sec.

YORKSHIRE BRANCH.

The Executive Committee have decided to continue the activities of the Branch so far as circumstances allow. Meetings are being held in or near Leeds. The President for 1940–1941 is E. R. Clarke, Aireborough Grammar School, and the Acting Honorary Secretary is H. H. Watts, 24 Batcliffe Mount, Leeds, 6.

BOOKS WANTED.

Will any member who is prepared to sell any of the following books please communicate with A. G. Champernowne, 8 Forres Gardens, London, N.W. 11.

Böhm, *Elliptische Funkt.*; Ed. Heis, *Hyperbolische Funkt.*; S. Gunther, *Die gew . . . hyperbolische Funkt.*; Doehlemann, *Geometrische Trans.* (2 vols., 1900-1907); König-Krafft, *Elliptische Funkt.*; Bacherer, *Vektoranalysis*; Leman, *Vom periodischen Dezimalbruch zur Zahlentheorie*; Bieberbach, *Anal. Geometrie* and *Funktionentheorie*; Lipschitz, *Analysis*; Kowalewski, *Anal. Geometrie* and *Lehrbuch der höheren Math.*; Bohnert, *Elementare Stereometrie*; Tannery, *Théorie des Fonctions d'une variable.*

BOOKS RECEIVED FOR REVIEW.

A. A. Albert. *Introduction to algebraic theories.* Pp. viii, 137. 10s. 6d. 1941. (University of Chicago Press; Cambridge University Press)

R. H. Baker. *An introduction to astronomy.* 2nd edition. Pp. 315. 12s. 6d. 1940. (Macmillan)

W. Briggs and **G. H. Bryan.** *The tutorial algebra. I. Intermediate course.* Fifth edition, revised and rewritten by G. Walker. Pp. xv, 457. 7s. 6d. 1940. (University Tutorial Press)

C. Burt. *The factors of the mind. An introduction to factor-analysis in psychology.* Pp. xiv, 509. 21s. 1940. (University of London Press)

A. J. Cook. *Geometry for today.* Based on A. W. Siddons and R. T. Hughes, *A junior geometry.* Pp. x, 260. $1. 1940. (Macmillan Company of Canada)

C. A. Coulson. *Waves. A mathematical account of the common types of wave motion.* Pp. xii, 156. 5s. 1941. University Mathematical Texts. (Oliver and Boyd)

T. Dantzig. *Number. The language of science.* 2nd edition. Pp. viii, 320. 10s. 1940. (Allen and Unwin)

Mukana Lal Dhawan. *Contributions to the fundamentals of arithmetic.* Pp. ii, 69. 7 annas. 1940. (Lahore)

W. L. Ferrar. *Algebra. A text-book of determinants, matrices and algebraic forms.* Pp. vii, 202. 12s. 6d. 1941. (Oxford)

H. Gupta. *Symmetric functions in the theory of integral numbers.* Pp. vii, 105. 1940. Lucknow University Studies, XIV. (University of Lucknow)

G. H. Hardy. *A mathematician's apology.* Pp. vii, 93. 3s. 6d. 1940. (Cambridge)

G. H. Hardy. *Ramanujan. Twelve lectures on subjects suggested by his life and work.* Pp. 236. 25s. 1940. (Cambridge)

W. V. D. Hodge. *The theory and application of harmonic integrals.* Pp. ix, 281. 21s. 1941. (Cambridge)

Sir James Jeans. *An introduction to the kinetic theory of gases.* Pp. 311. 15s. 1940. (Cambridge)

L. C. Karpinski. *Bibliography of mathematical works printed in America through 1850.* Pp. xxvi, 697. 6 dollars. 1940. (University of Michigan Press; Humphrey Milford)

G. W. Manfield. *Essential arithmetic. I. II. III.* Pp. 80 each. 1s. 3d. each. *Teachers' book.* Pp. 80. 5s. 1940. (Arnold)

R. R. Middlemiss. *Differential and integral calculus.* Pp. x, 416. 17s. 6d. 1940. (McGraw-Hill)

J. Murray. *The New Method arithmetics.* Pupil's book I. Pp. 95. 1s. 4d. Teacher's book I. Pp. 64. 2s. 6d. Pupil's book II. Pp. 95. 1s. 6d. Teacher's book II. Pp. 62. 2s. 9d. 1940. (University of London Press)

E. G. Phillips. *Functions of a complex variable with applications.* Pp. xi, 140. 4s. 6d. 1940. University Mathematical Texts. (Oliver and Boyd)

H. C. Plummer. *Four-figure tables with mathematical formulae.* Pp. iv, 89. 3s. 6d. 1941. (Macmillan)

A. S. Ramsey. *An introduction to the theory of Newtonian attraction.* Pp. ix, 184. 10s. 6d. 1940. (Cambridge)

G. de B. Robinson. *The foundations of geometry.* Pp. xi, 163. $2. 1940. Mathematical Expositions, 1. (University of Toronto Press)

J. A. Shohat, E. Hille and **J. L. Walsh.** *A bibliography on orthogonal polynomials.* Pp. ix, 204. $3. 1940. Bulletin of the National Research Council, 103. (National Academy of Sciences, Washington)

R. V. Southwell. *Relaxation methods in engineering science. A treatise on approximate computation.* Pp. vi, 252. 17s. 6d. 1940. (Oxford)

G. H. Thomson. *What are Moray House tests?* Pp. 8. 6d. 1940. (University of London Press)

L. Turner. *General mathematics. I.* Pp. viii, 238. 3s. 6d. 1941. (Arnold)

QUEENSLAND BRANCH.

Report for 1940–1941.

The last Annual Meeting was held on 24th May 1940. The Annual Report and the Financial Statement for the year then ending were presented to the Meeting and were adopted: the officers for the coming year were elected. The subject of Professor Simonds' Presidential Address was "The Newton-Leibniz Controversy".

During the year two General Meetings were held at the University. At the first, on 26th July 1940, Mr. E. W. Jones, B.A., read a paper on "Some Statistical Formulae", and at the second, on 1st November, Mr. A. W. Young, B.Sc., read a paper entitled "Geometry in Secondary Schools".

The number of members of the Branch at present is 29, of whom 10 are members of The Mathematical Association.

The Financial Statement shows a credit balance of £9 5s. 5d., which represents an increase of £1 8s. 5d. for the year.

Copies of the *Mathematical Gazette* are circulated as they come to hand.

The thanks of the Branch go to those who have provided papers during the year.

The Committee is as follows: *President*, Professor E. F. Simonds; *Vice-Presidents*, Mr. S. Stephenson, Mr. R. A. Kerr; *Hon. Secretary and Treasurer*, Mr. J. P. McCarthy; *Members*, Miss E. H. Raybould, Miss W. Hossack, Mr. E. W. Jones, Mr. I. Waddle, Mr. J. C. Deeney.

J. P. McCarthy, Hon. Sec.

BOOKS WANTED.

Will any member who is prepared to sell either of the following books please communicate with P. A. Moran, University Mathematical Laboratory, Cambridge:

H. Bateman: *Partial Differential Equations of Mathematical Physics* (Cambridge, 1932).

R. Courant und D. Hilbert: *Methoden der mathematischen Physik*, II (Berlin, 1937).

BOOKS RECEIVED FOR REVIEW.

R. C. Archibald. *Outline of the history of mathematics.* Fifth edition. Pp. 76. 75 cents, postpaid, remittance with order. 1941. (Mathematical Association of America, Oberlin, Ohio)

E. T. Bell. *The development of mathematics.* Pp. xiii, 583. 1940. 31s. 6d. (McGraw-Hill)

A. F. Buchan and **R. Borthwick.** *First stage mathematics, suitable for the R.A.F. and A.T.C.* Pp. 95. 2s. 1941. (University of London Press)

R. V. Churchill. *Fourier series and boundary value problems.* Pp. ix, 206. 17s. 6d. 1941. (McGraw-Hill)

R. W. Dull. *Mathematics for engineers.* Second edition. Pp. xviii, 780. 35s. 1941. (McGraw-Hill)

L. P. Eisenhart. *An introduction to differential geometry.* Pp. x, 304. 21s. 6d. 1941. (Princeton University Press; Oxford University Press)

H. G. Forder. *The calculus of extension.* Pp. xvi, 490. 30s. 1941. (Cambridge)

M. F. Headlam. *Sir Thomas Little Heath, K.C.B., K.C.V.O., F.R.S.*, 1861-1940. Pp. 16. 2s. 1941. Extracted from the *Proceedings of the British Academy.* (Oxford University Press)

J. Murray. *The New Method arithmetics.* Pupil's book 3. Pp. 95. Paper 1s. 5d.; limp cloth 1s. 7d. Teacher's book 3. Pp. 62. 2s. 9d. 1941. (University of London Press)

C. C. Peters and **W. R. Van Voorhis.** *Statistical procedures and their mathematical bases.* Pp. xiii, 516. 31s. 6d. 1940. (McGraw-Hill)

W. V. Quine. *Mathematical logic.* Pp. xiii, 348. 21s. 1940. (Norton, New York; George Allen and Unwin)

J. Trevelyan and **J. Morley.** *Functional arithmetic through citizenship. IV. One and All.* Pp. vi, 90. 1s. 9d. 1940. (Longmans, Green)

R. S. Underwood and **F. W. Sparks.** *Living mathematics.* Pp. ix, 365. 20s. 1940. (McGraw-Hill)

S. A. Walling and **J. C. Hill.** *Aircraft mathematics.* Pp. 189. 2s. 9d. 1941. (Cambridge University Press)

S. E. Wilson. *Decimal Hoppus tables, showing the solid contents of round timber in Hoppus feet and decimals.* Pp. 60. 6s. 1941. (Scientific Computing Service)

Arithmetic in junior schools. A report on the teaching of arithmetic and spatial knowledge in junior schools. Prepared by the Mathematics section of the Metropolitan Branch of the Training College Association. Pp. xvi, 76. 2s. 1940. (Longmans)

Barlow's tables of squares, cubes, square roots, cube roots and reciprocals of all integer numbers up to 12,500. Fourth edition. Edited by L. J. Comrie. Pp. xii, 258. 8s. 6d., postage 6d. 1941. (Spon)

British Association for the advancement of science. Mathematical tables. VIII. Number-divisor tables. Designed and in part prepared by J. W. L. Glaisher. Extended and edited by the Committee for the calculation of mathematical tables. Pp. x, 100. 15s. 1940. (Cambridge)

British Association for the advancement of science. Mathematical tables. IX. Tables of powers giving integral powers of integers. Initiated by J. W. L. Glaisher, extended by W. G. Bickley, C. E. Gwyther, J. C. P. Miller, E. J. Ternouth, on behalf of the Committee for the calculation of mathematical tables. Pp. xii, 132. 15s. 1940. (Cambridge)

Mathematics in general education. A report of the committee on the function of mathematics in general education, for the Commission on secondary school curriculum. Pp. xiv, 423. 15s. 1940. (Appleton-Century)

Facsimiles of two papers by Bayes. Prepared under the direction of W. E. Deming. $1. 1940. (Department of Agriculture, Washington)

The Nineteen Forty mental measurements yearbook. Edited by O. K. Buros. Pp. xxiii, 674. $6; orders sent directly, discount 10%. 1941. (Mental Measurements Yearbook, 32 Lincoln Avenue, Highland Park, New Jersey)

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

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Intending members are requested to communicate with one of the Secretaries, (G. L. Parsons, Merchant Taylors' School, Sandy Lodge, Northwood, Middlesex; Mrs. E. M. Williams, 17, Belgrave Square, Nottingham. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette".

Change of Address should be notified to Mrs. Williams. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

Subscriptions should be paid to the Hon. Treasurer, Mathematical Association, Gordon House, Winchcombe, Gloucestershire.

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